The Maths Workbook

Oxford University Department of Economics

The *Maths Workbook* has been developed for use by students preparing for the Preliminary Examination in PPE, E&M and History and Economics. It covers the mathematical techniques that students are expected to know for the exam.

The *Maths Workbook* was written by Margaret Stevens, with the help of: Alan Beggs, David Foster, Mary Gregory, Ben Irons, Godfrey Keller, Sujoy Mukerji, Mathan Satchi, Patrick Wallace, Tania Wilson.

Complementary Textbooks

As far as possible the *Workbook* is self-contained, but it should be used in conjunction with standard textbooks for a fuller coverage:

- Malcolm Pemberton and Nicholas Rao *Mathematics for Economists: An Introductory Textbook* 4th edition 2016, or earlier editions. (Comprehensive; appropriate level. This book is not included in the specific references in workbook chapters, but students should find it easy to identify relevant sections.)
- Ian Jacques *Mathematics for Economics and Business* 7th Edition 2013, or earlier editions. (The most elementary.)
- Martin Anthony and Norman Biggs *Mathematics for Economics and Finance*, 1996. (Useful and concise, but less suitable for students who have not previously studied mathematics to A-level.)
- Carl P. Simon and Lawrence Blume *Mathematics for Economists*, 1994 or 2010. (A good but more advanced textbook, that goes well beyond the *Workbook*.)
- Hal R. Varian *Intermediate Microeconomics: A Modern Approach* covers many of the economic applications, particularly in the Appendices to individual chapters, where calculus is used.
- In addition, students who have not studied A-level maths, or feel that their maths is weak, may find it helpful to use one of the many excellent textbooks available for A-level Pure Mathematics (particularly the first three modules).

How to Use the Workbook

There are ten chapters, each of which can be used as the basis for a class. It is intended that students should be able to work through each chapter alone, doing the exercises and checking their own answers. References to some of the textbooks listed above are given at the end of each section.

At the end of each chapter is a worksheet, the answers for which are available for the use of tutors only.

The first two chapters are intended mainly for students who have not done A-level maths: they assume GCSE maths only. In subsequent chapters, students who have done A-level will find both familiar and new material.

Contents

Errors and Improvements

The *Workbook* was completed in summer 2003. As far as possible, examples, exercises and answers were carefully checked. Since then, a small number of errors have been corrected, and updated versions of the chapters placed on the Economics Department Weblearn site. Versions of each chapter are identified by the release date, which can be found on the page containing the answers to the exercises (in most cases, the penultimate page).

Please report any mistakes that you notice, however small or large. Suggestions from tutors and students for improvements to the *Workbook* are very welcome.

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CHAPTER 1

Review of Algebra

Much of the material in this chapter is revision from GCSE maths (although some of the exercises are harder). Some of it – particularly the work on logarithms – may be new if you have not done A-level maths. If you have done A-level, and are confident, you can skip most of the exercises and just do the worksheet, using the chapter for reference where necessary.

1. Algebraic Expressions

1.1. Evaluating Algebraic Expressions

EXAMPLES 1.1:

(i) A firm that manufactures widgets has m machines and employs n workers. The number of widgets it produces each day is given by the expression $m^2(n-3)$. How many widgets does it produce when $m = 5$ and $n = 6$?

$$\text{Number of widgets} = 5^2 \times (6-3) = 25 \times 3 = 75$$

(ii) In another firm, the cost of producing x widgets is given by $3x^2 + 5x + 4$. What is the cost of producing (a) 10 widgets (b) 1 widget?

$$\begin{aligned} \text{When } x = 10, \text{ cost} &= (3 \times 10^2) + (5 \times 10) + 4 = 300 + 50 + 4 = 354 \\ \text{When } x = 1, \text{ cost} &= 3 \times 1^2 + 5 \times 1 + 4 = 3 + 5 + 4 = 12 \end{aligned}$$

It might be clearer to use brackets here, but they are not essential: the rule is that $\times$ and $\div$ are evaluated before $+$ and $-$.

(iii) Evaluate the expression $8y^4 - \frac{12}{6-y}$ when $y = -2$.
(Remember that y^4 means $y \times y \times y \times y$.)

$$8y^4 - \frac{12}{6-y} = 8 \times (-2)^4 - \frac{12}{6-(-2)} = 8 \times 16 - \frac{12}{8} = 128 - 1.5 = 126.5$$

(If you are uncertain about using negative numbers, work through *Jacques* pp.7–9.)

EXERCISES 1.1: Evaluate the following expressions when $x = 1$, $y = 3$, $z = -2$ and $t = 0$:
(a) $3y^2 - z$ (b) $xt + z^3$ (c) $(x + 3z)y$ (d) $\frac{y}{z} + \frac{2}{x}$ (e) $(x + y)^3$ (f) $5 - \frac{x+3}{2t-z}$

1.2. Manipulating and Simplifying Algebraic Expressions

EXAMPLES 1.2:

(i) Simplify $1 + 3x - 4y + 3xy + 5y^2 + y - y^2 + 4xy - 8$.
This is done by *collecting like terms*, and adding them together:

$$\begin{aligned} & 1 + 3x - 4y + 3xy + 5y^2 + y - y^2 + 4xy - 8 \\ = \;& 5y^2 - y^2 + 3xy + 4xy + 3x - 4y + y + 1 - 8 \\ = \;& 4y^2 + 7xy + 3x - 3y - 7 \end{aligned}$$

The order of the terms in the answer doesn't matter, but we often put a positive term first, and/or write "higher-order" terms such as y^2 before "lower-order" ones such as y or a number.

(ii) Simplify $5(x - 3) - 2x(x + y - 1)$.
Here we need to *multiply out the brackets* first, and then collect terms:

$$\begin{aligned} 5(x-3) - 2x(x+y-1) &= 5x - 15 - 2x^2 - 2xy + 2x \\ &= 7x - 2x^2 - 2xy - 15 \end{aligned}$$

(iii) Multiply x^3 by x^2.

$$x^3 \times x^2 = x \times x \times x \times x \times x = x^5$$

(iv) Divide x^3 by x^2.
We can write this as a fraction, and cancel:

$$x^3 \div x^2 = \frac{x \times x \times x}{x \times x} = \frac{x}{1} = x$$

(v) Multiply $5x^2y^4$ by $4yx^6$.

$$\begin{aligned} 5x^2y^4 \times 4yx^6 &= 5 \times x^2 \times y^4 \times 4 \times y \times x^6 \\ &= 20 \times x^8 \times y^5 \\ &= 20x^8y^5 \end{aligned}$$

Note that you can always change the order of multiplication.

(vi) Divide $6x^2y^3$ by $2yx^5$.

$$\begin{aligned} 6x^2y^3 \div 2yx^5 &= \frac{6x^2y^3}{2yx^5} = \frac{3x^2y^3}{yx^5} = \frac{3y^3}{yx^3} \\ &= \frac{3y^2}{x^3} \end{aligned}$$

(vii) Add $\frac{3x}{y}$ and $\frac{y}{2}$.
The rules for algebraic fractions are just the same as for numbers, so here we find a *common denominator*:

$$\begin{aligned} \frac{3x}{y} + \frac{y}{2} &= \frac{6x}{2y} + \frac{y^2}{2y} \\ &= \frac{6x + y^2}{2y} \end{aligned}$$

(viii) Divide $\frac{3x^2}{y}$ by $\frac{xy^3}{2}$.

$$\begin{aligned}\frac{3x^2}{y} \div \frac{xy^3}{2} &= \frac{3x^2}{y} \times \frac{2}{xy^3} = \frac{3x^2 \times 2}{y \times xy^3} = \frac{6x^2}{xy^4} \\ &= \frac{6x}{y^4}\end{aligned}$$

EXERCISES 1.2: Simplify the following as much as possible:

(1) (a) $3x - 17 + x^3 + 10x - 8$ (b) $2(x + 3y) - 2(x + 7y - x^2)$

(2) (a) $z^2x - (z + 1) + z(2xz + 3)$ (b) $(x + 2)(x + 4) + (3 - x)(x + 2)$

(3) (a) $\dfrac{3x^2y}{6x}$ (b) $\dfrac{12xy^3}{2x^2y^2}$

(4) (a) $2x^2 \div 8xy$ (b) $4xy \times 5x^2y^3$

(5) (a) $\dfrac{2x}{y} \times \dfrac{y^2}{2x}$ (b) $\dfrac{2x}{y} \div \dfrac{y^2}{2x}$

(6) (a) $\dfrac{2x+1}{4} + \dfrac{x}{3}$ (b) $\dfrac{1}{x-1} - \dfrac{1}{x+1}$ (giving the answers as a single fraction)

1.3. Factorising

A number can be written as the product of its factors. For example: $30 = 5 \times 6 = 5 \times 3 \times 2$. Similarly "factorise" an algebraic expression means "write the expression as the product of two (or more) expressions." Of course, some numbers (primes) don't have any proper factors, and similarly, some algebraic expressions can't be factorised.

EXAMPLES 1.3:

(i) Factorise $6x^2 + 15x$.
Here, $3x$ is a common factor of each term in the expression so:

$$6x^2 + 15x = 3x(2x + 5)$$

The factors are $3x$ and $(2x + 5)$. You can check the answer by multiplying out the brackets.

(ii) Factorise $x^2 + 2xy + 3x + 6y$.
There is no common factor of all the terms but the first pair have a common factor, and so do the second pair, and this leads us to the factors of the whole expression:

$$\begin{aligned}x^2 + 2xy + 3x + 6y &= x(x + 2y) + 3(x + 2y) \\ &= (x + 3)(x + 2y)\end{aligned}$$

Again, check by multiplying out the brackets.

(iii) Factorise $x^2 + 2xy + 3x + 3y$.
We can try the method of the previous example, but it doesn't work. The expression can't be factorised.

(iv) Simplify $5(x^2+6x+3)-3(x^2+4x+5)$.
Here we can first multiply out the brackets, then collect like terms, then factorise:

$$\begin{aligned} 5(x^2+6x+3)-3(x^2+4x+5) &= 5x^2+30x+15-3x^2-12x-15 \\ &= 2x^2+18x \\ &= 2x(x+9) \end{aligned}$$

EXERCISES 1.3: **Factorising**

(1) Factorise: (a) $3x+6xy$ (b) $2y^2+7y$ (c) $6a+3b+9c$

(2) Simplify and factorise: (a) $x(x^2+8)+2x^2(x-5)-8x$ (b) $a(b+c)-b(a+c)$

(3) Factorise: $xy+2y+2xz+4z$

(4) Simplify and factorise: $3x(x+\frac{4}{x})-4(x^2+3)+2x$

1.4. Polynomials

Expressions such as

$$5x^2-9x^4-20x+7 \text{ and } 2y^5+y^3-100y^2+1$$

are called *polynomials*. A polynomial in x is a sum of terms, and each term is either a power of x (multiplied by a number called a *coefficient*), or just a number known as a *constant*. All the powers must be positive integers. (Remember: an *integer* is a positive or negative whole number.) The *degree* of the polynomial is the highest power. A polynomial of degree 2 is called a *quadratic* polynomial.

EXAMPLES 1.4: *Polynomials*

(i) $5x^2-9x^4-20x+7$ is a polynomial of degree 4. In this polynomial, the coefficient of x^2 is 5 and the coefficient of x is -20. The constant term is 7.

(ii) x^2+5x+6 is a quadratic polynomial. Here the coefficient of x^2 is 1.

1.5. Factorising Quadratics

In section 1.3 we factorised a quadratic polynomial by finding a common factor of each term: $6x^2+15x=3x(2x+5)$. But this only works because there is no constant term. Otherwise, we can try a different method:

EXAMPLES 1.5: *Factorising Quadratics*

(i) x^2+5x+6

- Look for two numbers that multiply to give 6, and add to give 5:

$$2\times 3=6 \text{ and } 2+3=5$$

- Split the "x"-term into two:

$$x^2+2x+3x+6$$

- Factorise the first pair of terms, and the second pair:

$$x(x+2)+3(x+2)$$

- $(x+2)$ is a factor of both terms so we can rewrite this as:

$$(x+3)(x+2)$$

- So we have:

$$x^2+5x+6=(x+3)(x+2)$$

(ii) y^2-y-12
In this example the two numbers we need are 3 and -4, because $3\times(-4)=-12$ and $3+(-4)=-1$. Hence:

$$\begin{aligned} y^2-y-12 &= y^2+3y-4y-12 \\ &= y(y+3)-4(y+3) \\ &= (y-4)(y+3) \end{aligned}$$

(iii) $2x^2-5x-12$
This example is slightly different because the coefficient of x^2 is not 1.

- Start by multiplying together the coefficient of x^2 and the constant:

$$2\times(-12)=-24$$

- Find two numbers that multiply to give -24, and add to give -5.

$$3\times(-8)=-24 \text{ and } 3+(-8)=-5$$

- Proceed as before:

$$\begin{aligned} 2x^2-5x-12 &= 2x^2+3x-8x-12 \\ &= x(2x+3)-4(2x+3) \\ &= (x-4)(2x+3) \end{aligned}$$

(iv) x^2+x-1
The method doesn't work for this example, because we can't see any numbers that multiply to give -1, but add to give 1. (In fact there is a pair of numbers that does so, but they are not integers so we are unlikely to find them.)

(v) x^2-49
The two numbers must multiply to give -49 and add to give zero. So they are 7 and -7:

$$\begin{aligned} x^2-49 &= x^2+7x-7x-49 \\ &= x(x+7)-7(x+7) \\ &= (x-7)(x+7) \end{aligned}$$

The last example is a special case of the result known as "the difference of two squares". If a and b are any two numbers:

$$a^2-b^2=(a-b)(a+b)$$

EXERCISES 1.4: Use the method above (if possible) to factorise the following quadratics:

(1) x^2+4x+3
(2) $y^2+10-7y$
(3) $2x^2+7x+3$
(4) $z^2+2z-15$
(5) $4x^2-9$
(6) $y^2-10y+25$
(7) x^2+3x+1

1.6. Rational Numbers, Irrational Numbers, and Square Roots

A *rational* number is a number that can be written in the form $\frac{p}{q}$ where p and q are integers. An *irrational* number is a number that is not rational. It can be shown that if a number can be written as a terminating decimal (such as 1.32) or a recurring decimal (such as 3.7425252525...) then it is rational. Any decimal that does not terminate or recur is irrational.

EXAMPLES 1.6: *Rational and Irrational Numbers*

(i) 3.25 is rational because $3.25 = 3\frac{1}{4} = \frac{13}{4}$.

(ii) -8 is rational because $-8 = \frac{-8}{1}$. Obviously, all integers are rational.

(iii) To show that 0.12121212... is rational check on a calculator that it is equal to $\frac{4}{33}$.

(iv) $\sqrt{2} = 1.41421356237...$ is irrational.

Most, but not all, square roots are irrational:

EXAMPLES 1.7: *Square Roots*

(i) (Using a calculator) $\sqrt{5} = 2.2360679774...$ and $\sqrt{12} = 3.4641016151...$

(ii) $5^2 = 25$, so $\sqrt{25} = 5$

(iii) $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$, so $\sqrt{\frac{4}{9}} = \frac{2}{3}$

Rules for Square Roots:

$$\sqrt{ab} = \sqrt{a}\sqrt{b} \qquad \text{and} \qquad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

EXAMPLES 1.8: *Using the rules to manipulate expressions involving square roots*

(i) $\sqrt{2} \times \sqrt{50} = \sqrt{2 \times 50} = \sqrt{100} = 10$

(ii) $\sqrt{48} = \sqrt{16}\sqrt{3} = 4\sqrt{3}$

(iii) $\frac{\sqrt{98}}{\sqrt{8}} = \sqrt{\frac{98}{8}} = \sqrt{\frac{49}{4}} = \frac{\sqrt{49}}{\sqrt{4}} = \frac{7}{2}$

(iv) $\frac{-2+\sqrt{20}}{2} = -1 + \frac{\sqrt{20}}{2} = -1 + \frac{\sqrt{5}\sqrt{4}}{2} = -1 + \sqrt{5}$

(v) $\frac{8}{\sqrt{2}} = \frac{8\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}} = \frac{8\sqrt{2}}{2} = 4\sqrt{2}$

(vi) $\frac{\sqrt{27y}}{\sqrt{3y}} = \sqrt{\frac{27y}{3y}} = \sqrt{9} = 3$

(vii) $\sqrt{x^3y}\sqrt{4xy} = \sqrt{x^3y \times 4xy} = \sqrt{4x^4y^2} = \sqrt{4}\sqrt{x^4}\sqrt{y^2} = 2x^2y$

EXERCISES 1.5: **Square Roots**

(1) Show that: (a) $\sqrt{2} \times \sqrt{18} = 6$ (b) $\sqrt{245} = 7\sqrt{5}$ (c) $\frac{15}{\sqrt{3}} = 5\sqrt{3}$

(2) Simplify: (a) $\frac{\sqrt{45}}{3}$ (b) $\sqrt{2x^3} \times \sqrt{8x}$ (c) $\sqrt{2x^3} \div \sqrt{8x}$ (d) $\frac{1}{3}\sqrt{18y^2}$

Further reading and exercises

- *Jacques* §1.4 has lots more practice of algebra. If you have had any difficulty with the work so far, you should work through it before proceeding.

2. Indices and Logarithms

2.1. Indices

We know that x^3 means $x \times x \times x$. More generally, if n is a positive integer, x^n means "x multiplied by itself n times". We say that x is *raised to the power* n. Alternatively, n may be described as the *index* of x in the expression x^n.

EXAMPLES 2.1:

(i) $5^4 \times 5^3 = 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^7$.

(ii) $\dfrac{x^5}{x^2} = \dfrac{x \times x \times x \times x \times x}{x \times x} = x \times x \times x = x^3$.

(iii) $\left(y^3\right)^2 = y^3 \times y^3 = y^6$.

Each of the above examples is a special case of the general rules:

- $a^m \times a^n = a^{m+n}$
- $\dfrac{a^m}{a^n} = a^{m-n}$
- $(a^m)^n = a^{m \times n}$

Now, a^n also has a meaning when n is zero, or negative, or a fraction. Think about the second rule above. If $m = n$, this rule says:

$$a^0 = \frac{a^n}{a^n} = 1$$

If $m = 0$ the rule says:

$$a^{-n} = \frac{1}{a^n}$$

Then think about the third rule. If, for example, $m = \frac{1}{2}$ and $n = 2$, this rule says:

$$\left(a^{\frac{1}{2}}\right)^2 = a$$

which means that

$$a^{\frac{1}{2}} = \sqrt{a}$$

Similarly $a^{\frac{1}{3}}$ is the cube root of a, and more generally $a^{\frac{1}{n}}$ is the n^{th} root of a:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Applying the third rule above, we find for more general fractions:

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

We can summarize the rules for zero, negative, and fractional powers:

- $a^0 = 1$ (if $a \neq 0$)
- $a^{-n} = \dfrac{1}{a^n}$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$
- $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$

There are two other useful rules, which may be obvious to you. If not, check them using some particular examples:

- $a^n b^n = (ab)^n$ and - $\dfrac{a^n}{b^n} = \left(\dfrac{a}{b}\right)^n$

EXAMPLES 2.2: *Using the Rules for Indices*

(i) $3^2 \times 3^3 = 3^5 = 243$

(ii) $\left(5^2\right)^{\frac{1}{2}} = 5^{2\times\frac{1}{2}} = 5$

(iii) $4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = 2^3 = 8$

(iv) $36^{-\frac{3}{2}} = \left(36^{\frac{1}{2}}\right)^{-3} = 6^{-3} = \dfrac{1}{6^3} = \dfrac{1}{216}$

(v) $\left(3\frac{3}{8}\right)^{\frac{2}{3}} = \left(\dfrac{27}{8}\right)^{\frac{2}{3}} = \dfrac{27^{\frac{2}{3}}}{8^{\frac{2}{3}}} = \dfrac{\left(27^{\frac{1}{3}}\right)^2}{\left(8^{\frac{1}{3}}\right)^2} = \dfrac{3^2}{2^2} = \dfrac{9}{4}$

2.2. Logarithms

You can think of *logarithm* as another word for index or power. To define a logarithm we first choose a particular *base*. Your calculator probably uses base 10, but we can take any positive integer, a. Now take any positive number, x.

The logarithm of x to the base a is:
the power to which the base must be raised to obtain x.
If $x = a^n$ then $\log_a x = n$

In fact the statement: $\log_a x = n$ is simply another way of saying: $x = a^n$. Note that, since a^n is positive for all values of n, there is no such thing as the log of zero or a negative number.

EXAMPLES 2.3:

(i) Since we know $2^5 = 32$, we can say that the log of 32 to the base 2 is 5: $\log_2 32 = 5$

(ii) From $3^4 = 81$ we can say $\log_3 81 = 4$

(iii) From $10^{-2} = 0.01$ we can say $\log_{10} 0.01 = -2$

(iv) From $9^{\frac{1}{2}} = 3$ we can say $\log_9 3 = 0.5$

(v) From $a^0 = 1$, we can say that the log of 1 to *any* base is zero: $\log_a 1 = 0$

(vi) From $a^1 = a$, we can say that for any base a, the log of a is 1: $\log_a a = 1$

Except for easy examples like these, you cannot calculate logarithms of particular numbers in your head. For example, if you wanted to know the logarithm to base 10 of 3.4, you would need to find out what power of 10 is equal to 3.4, which is not easy. So instead, you can use your calculator. Check the following examples of logs to base 10:

EXAMPLES 2.4: Using a calculator we find that (correct to 5 decimal places):

(i) $\log_{10} 3.4 = 0.53148$ (ii) $\log_{10} 125 = 2.09691$ (iii) $\log_{10} 0.07 = -1.15490$

There is a way of calculating logs to other bases, using logs to base 10. But the only other base that you really need is the special base e, which we will meet later.

2.3. Rules for Logarithms

Since logarithms are powers, or indices, there are rules for logarithms which are derived from the rules for indices in section 2.1:

- $\log_a xy = \log_a x + \log_a y$
- $\log_a \dfrac{x}{y} = \log_a x - \log_a y$
- $\log_a x^b = b \log_a x$
- $\log_a a = 1$
- $\log_a 1 = 0$

To see where the first rule comes from, suppose: $m = \log_a x$ and $n = \log_a y$
This is equivalent to: $x = a^m$ and $y = a^n$
Using the first rule for indices: $xy = a^m a^n = a^{m+n}$
But this means that: $\log_a xy = m + n = \log_a x + \log_a y$
which is the first rule for logs.
You could try proving the other rules similarly.

Before electronic calculators were available, printed tables of logs were used calculate, for example, $14.58 \div 0.3456$. You could find the log of each number in the tables, then (applying the second rule) subtract them, and use the tables to find the "anti-log" of the answer.

EXAMPLES 2.5: *Using the Rules for Logarithms*

(i) Express $2\log_a 5 + \frac{1}{3}\log_a 8$ as a single logarithm.

$$\begin{aligned} 2\log_a 5 + \tfrac{1}{3}\log_a 8 &= \log_a 5^2 + \log_a 8^{\frac{1}{3}} = \log_a 25 + \log_a 2 \\ &= \log_a 50 \end{aligned}$$

(ii) Express $\log_a\left(\frac{x^2}{y^3}\right)$ in terms of $\log x$ and $\log y$.

$$\begin{aligned} \log_a\left(\tfrac{x^2}{y^3}\right) &= \log_a x^2 - \log_a y^3 \\ &= 2\log_a x - 3\log_a y \end{aligned}$$

EXERCISES 1.6: **Indices and Logarithms**

(1) Evaluate (without a calculator):
(a) $64^{\frac{2}{3}}$ (b) $\log_2 64$ (c) $\log_{10} 1000$ (d) $4^{130} \div 4^{131}$

(2) Simplify: (a) $2x^5 \times x^6$ (b) $\dfrac{(xy)^2}{x^3y^2}$ (c) $\log_{10}(xy) - \log_{10} x$ (d) $\log_{10}(x^3) \div \log_{10} x$

(3) Simplify: (a) $\left(3\sqrt{ab}\right)^6$ (b) $\log_{10} a^2 + \frac{1}{3}\log_{10} b - 2\log_{10} ab$

Further reading and exercises

- *Jacques* §2.3 covers all the material in section 2, and provides more exercises.

3. Solving Equations

3.1. Linear Equations

Suppose we have an equation:

$$5(x-6) = x+2$$

Solving this equation means finding the value of x that makes the equation true. (Some equations have several, or many, solutions; this one has only one.)

To solve this sort of equation, we manipulate it by "doing the same thing to both sides." The aim is to get the variable x on one side, and everything else on the other.

EXAMPLES 3.1: Solve the following equations:

(i) $5(x-6) = x+2$

Remove brackets:	$5x-30$	$=$	$x+2$
$-x$ from both sides:	$5x-x-30$	$=$	$x-x+2$
Collect terms:	$4x-30$	$=$	2
$+30$ to both sides:	$4x$	$=$	32
$\div$ both sides by 4:	x	$=$	8

(ii) $\dfrac{5-x}{3} + 1 = 2x+4$

Here it is a good idea to remove the fraction first:

$\times$ all terms by 3:	$5-x+3$	$=$	$6x+12$
Collect terms:	$8-x$	$=$	$6x+12$
$-6x$ from both sides:	$8-7x$	$=$	12
-8 from both sides:	$-7x$	$=$	4
$\div$ both sides by -7:	x	$=$	$-\frac{4}{7}$

(iii) $\dfrac{5x}{2x-9} = 1$

Again, remove the fraction first:

$\times$ by $(2x-9)$:	$5x$	$=$	$2x-9$
$-2x$ from both sides:	$3x$	$=$	-9
$\div$ both sides by 3:	x	$=$	-3

All of these are linear equations: once we have removed the brackets and fractions, each term is either an x-term or a constant.

EXERCISES 1.7: Solve the following equations:

(1) $5x+4=19$

(2) $2(4-y) = y+17$

(3) $\frac{2x+1}{5} + x - 3 = 0$

(4) $2 - \frac{4-z}{z} = 7$

(5) $\frac{1}{4}(3a+5) = \frac{3}{2}(a+1)$

3.2. Equations involving Parameters

Suppose x satisfies the equation: $5(x-a) = 3x+1$

Here a is a *parameter*: a letter representing an unspecified number. The solution of the equation will depend on the value of a. For example, you can check that if $a = 1$, the solution is $x = 3$, and if $a = 2$ the solution is $x = 5.5$.

Without knowing the value of a, we can still solve the equation for x, to find out exactly how x depends on a. As before, we manipulate the equation to get x on one side and everything else on the other:

$$\begin{aligned} 5x - 5a &= 3x + 1 \\ 2x - 5a &= 1 \\ 2x &= 5a + 1 \\ x &= \frac{5a+1}{2} \end{aligned}$$

We have obtained the solution for x in terms of the parameter a.

EXERCISES 1.8: **Equations involving parameters**

(1) Solve the equation $ax + 4 = 10$ for x.

(2) Solve the equation $\frac{1}{2}y + 5b = 3b$ for y.

(3) Solve the equation $2z - a = b$ for z.

3.3. Changing the Subject of a Formula

$V = \pi r^2 h$ is the formula for the volume of a cylinder with radius r and height h - so if you know r and h, you can calculate V. We could rearrange the formula to *make r the subject*:

Write the equation as:	$\pi r^2 h$	$=$	V
Divide by πh:	r^2	$=$	$\frac{V}{\pi h}$
Square root both sides:	r	$=$	$\sqrt{\frac{V}{\pi h}}$

This gives us a formula for r in terms of V and h. The procedure is exactly the same as solving the equation for r.

EXERCISES 1.9: **Formulae and Equations**

(1) Make t the subject of the formula $v = u + at$

(2) Make a the subject of the formula $c = \sqrt{a^2 + b^2}$

(3) When the price of an umbrella is p, and daily rainfall is r, the number of umbrellas sold is given by the formula: $n = 200r - \frac{p}{6}$. Find the formula for the price in terms of the rainfall and the number sold.

(4) If a firm that manufuctures widgets has m machines and employs n workers, the number of widgets it produces each day is given by the formula $W = m^2(n-3)$. Find a formula for the number of workers it needs, if it has m machines and wants to produce W widgets.

3.4. Quadratic Equations

A quadratic equation is one that, once brackets and fractions have removed, contains terms in x^2, as well as (possibly) x-terms and constants. A quadratic equation can be rearranged to have the form:

$$ax^2 + bx + c = 0$$

where a, b and c are numbers and $a \neq 0$.

A simple quadratic equation is:

$$x^2 = 25$$

You can see immediately that $x = 5$ is a solution, but note that $x = -5$ satisfies the equation too. There are two solutions:

$$x = 5 \quad \text{and} \quad x = -5$$

Quadratic equations have either two solutions, or one solution, or no solutions. The solutions are also known as the *roots* of the equation. There are two general methods for solving quadratics; we will apply them to the example:

$$x^2 + 5x + 6 = 0$$

Method 1: Quadratic Factorisation
We saw in section 1.5 that the quadratic polynomial $x^2 + 5x + 6$ can be factorised, so we can write the equation as:

$$(x + 3)(x + 2) = 0$$

But if the product of two expressions is zero, this means that one of them must be zero, so we can say:

$$\begin{aligned} \text{either } x + 3 = 0 \quad &\Rightarrow \quad x = -3 \\ \text{or } x + 2 = 0 \quad &\Rightarrow \quad x = -2 \end{aligned}$$

The equation has two solutions, -3 and -2. You can check that these are solutions by substituting them back into the original equation.

Method 2: The Quadratic Formula
If the equation $ax^2 + bx + c = 0$ can't be factorised (or if it can, but you can't see how) you can use[1]:

$$\text{The Quadratic Formula:} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The notation $\pm$ indicates that an expression may take either a positive or negative value. So this is a formula for the two solutions $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$.

In the equation $x^2 + 5x + 6 = 0$, $a = 1$, $b = 5$ and $c = 6$. The quadratic formula gives us:

$$\begin{aligned} x &= \frac{-5 \pm \sqrt{5^2 - 4 \times 1 \times 6}}{2 \times 1} \\ &= \frac{-5 \pm \sqrt{25 - 24}}{2} \\ &= \frac{-5 \pm 1}{2} \end{aligned}$$

[1] *Antony & Biggs* §2.4 explains where the formula comes from.

So the two solutions are:

$$x = \frac{-5+1}{2} = -2 \text{ and } x = \frac{-5-1}{2} = -3$$

Note that in the quadratic formula $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, $b^2 - 4ac$ could turn out to be zero, in which case there is only one solution. Or it could be negative, in which case there are no solutions since we can't take the square root of a negative number.

Examples 3.2: Solve, if possible, the following quadratic equations.

(i) $x^2 + 3x - 10 = 0$
Factorise:

$$(x+5)(x-2) = 0$$

$$\Rightarrow x = -5 \text{ or } x = 2$$

(ii) $x(7-2x) = 6$
First, rearrange the equation to get it into the usual form:

$$\begin{aligned} 7x - 2x^2 &= 6 \\ -2x^2 + 7x - 6 &= 0 \\ 2x^2 - 7x + 6 &= 0 \end{aligned}$$

Now, we can factorise, to obtain:

$$(2x-3)(x-2) = 0$$

$$\begin{aligned} \text{either } 2x - 3 = 0 \quad &\Rightarrow \quad x = \tfrac{3}{2} \\ \text{or } x - 2 = 0 \quad &\Rightarrow \quad x = 2 \end{aligned}$$

The solutions are $x = \frac{3}{2}$ and $x = 2$.

(iii) $y^2 + 4y + 4 = 0$
Factorise:

$$\begin{aligned} (y+2)(y+2) &= 0 \\ \Longrightarrow y + 2 &= 0 \quad \Rightarrow y = -2 \end{aligned}$$

Therefore $y = -2$ is the only solution. (Or we sometimes say that the equation has a *repeated root* – the two solutions are the same.)

(iv) $x^2 + x - 1 = 0$
In section 1.5 we couldn't find the factors for this example. So apply the formula, putting $a = 1$, $b = 1$, $c = -1$:

$$x = \frac{-1 \pm \sqrt{1-(-4)}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

The two solutions, correct to 3 decimal places, are:

$$x = \tfrac{-1+\sqrt{5}}{2} = 0.618 \quad \text{and} \quad x = \tfrac{-1-\sqrt{5}}{2} = -1.618$$

Note that this means that the factors are, approximately, $(x - 0.618)$ and $(x + 1.618)$.

(v) $2z^2 + 2z + 5 = 0$

Applying the formula gives: $z = \dfrac{-2 \pm \sqrt{-36}}{4}$

So there are no solutions, because this contains the square root of a negative number.

(vi) $6x^2 + 2kx = 0$ (solve for x, treating k as a parameter)

Factorising:

$$2x(3x + k) = 0$$

$$\begin{aligned} \text{either } 2x = 0 \quad &\Rightarrow \quad x = 0 \\ \text{or } 3x + k = 0 \quad &\Rightarrow \quad x = -\frac{k}{3} \end{aligned}$$

EXERCISES 1.10: Solve the following quadratic equations, where possible:

(1) $x^2 + 3x - 13 = 0$

(2) $4y^2 + 9 = 12y$

(3) $3z^2 - 2z - 8 = 0$

(4) $7x - 2 = 2x^2$

(5) $y^2 + 3y + 8 = 0$

(6) $x(2x - 1) = 2(3x - 2)$

(7) $x^2 - 6kx + 9k^2 = 0$ (where k is a parameter)

(8) $y^2 - 2my + 1 = 0$ (where m is a parameter)
Are there any values of m for which this equation has no solution?

3.5. Equations involving Indices

EXAMPLES 3.3:

(i) $7^{2x+1} = 8$

Here the variable we want to find, x, appears in a power.
This type of equation can by solved by *taking logs of both sides*:

$$\begin{aligned} \log_{10}\left(7^{2x+1}\right) &= \log_{10}(8) \\ (2x+1)\log_{10} 7 &= \log_{10} 8 \\ 2x + 1 &= \frac{\log_{10} 8}{\log_{10} 7} = 1.0686 \\ 2x &= 0.0686 \\ x &= 0.0343 \end{aligned}$$

(ii) $(2x)^{0.65} + 1 = 6$

We can use the rules for indices to manipulate this equation:

$$
\begin{aligned}
&\text{Subtract 1 from both sides:} & (2x)^{0.65} &= 5 \\
&\text{Raise both sides to the power } \tfrac{1}{0.65}\text{:} & \left((2x)^{0.65}\right)^{\frac{1}{0.65}} &= 5^{\frac{1}{0.65}} \\
& & 2x &= 5^{\frac{1}{0.65}} = 11.894 \\
&\text{Divide by 2:} & x &= 5.947
\end{aligned}
$$

3.6. Equations involving Logarithms

EXAMPLES 3.4: Solve the following equations:

(i) $\log_5(3x-2) = 2$
From the definition of a logarithm, this equation is equivalent to:

$$3x - 2 = 5^2$$

which can be solved easily:

$$3x - 2 = 25 \Rightarrow x = 9$$

(ii) $10\log_{10}(5x+1) = 17$

$$
\begin{aligned}
\Rightarrow \log_{10}(5x+1) &= 1.7 \\
5x + 1 &= 10^{1.7} = 50.1187 \text{ (correct to 4 decimal places)} \\
x &= 9.8237
\end{aligned}
$$

EXERCISES 1.11: Solve the following equations:

(1) $\log_4(2+x) = 2$

(2) $16 = 5^{3t}$

(3) $2 + x^{0.4} = 8$

The remaining questions are a bit harder – skip them if you found this section difficult.

(4) $4.1 + 5x^{0.42} = 7.8$

(5) $6^{x^2-7} = 36$

(6) $\log_2(y^2+4) = 3$

(7) $3^{n+1} = 2^n$

(8) $2\log_{10}(x-2) = \log_{10}(x)$

Further reading and exercises

- For more practice on solving all the types of equation in this section, you could use an A-level pure maths textbook.
- *Jacques* §1.5 gives more detail on *Changing the Subject of a Formula*
- *Jacques* §2.1 and *Anthony & Biggs* §2.4 both cover the Quadratic Formula for *Solving Quadratic Equations*
- *Jacques* §2.3 has more *Equations involving Indices*

4. Simultaneous Equations

So far we have looked at equations involving one variable (such as x). An equation involving two variables, x and y, such as $x + y = 20$, has lots of solutions – there are lots of pairs of numbers x and y that satisfy it (for example $x = 3$ and $y = 17$, or $x = -0.5$ and $y = 20.5$).

But suppose we have two equations and two variables:

(1) $x + y = 20$

(2) $3x = 2y - 5$

There is just one pair of numbers x and y that satisfy both equations.

Solving a pair of simultaneous equations means finding the pair(s) of values that satisfy both equations. There are two approaches; in both the aim is to eliminate one of the variables, so that you can solve an equation involving one variable only.

Method 1: Substitution

Make one variable the subject of one of the equations (it doesn't matter which), and substitute it in the other equation.

From equation (1): $x = 20 - y$

Substitute for x in equation (2): $3(20 - y) = 2y - 5$

Solve for y:

$$\begin{aligned} 60 - 3y &= 2y - 5 \\ -5y &= -65 \\ y &= 13 \end{aligned}$$

From the equation in the first step: $x = 20 - 13 = 7$

The solution is $x = 7, y = 13$.

Method 2: Elimination

Rearrange the equations so that you can add or subtract them to eliminate one of the variables.

Write the equations as:

$$\begin{aligned} x + y &= 20 \\ 3x - 2y &= -5 \end{aligned}$$

Multiply the first one by 2:

$$\begin{aligned} 2x + 2y &= 40 \\ 3x - 2y &= -5 \end{aligned}$$

Add the equations together: $5x = 35 \quad \Rightarrow x = 7$

Substitute back in equation (1): $7 + y = 20 \quad \Rightarrow y = 13$

EXAMPLES 4.1: *Simultaneous Equations*

(i) Solve the equations $3x + 5y = 12$ and $2x - 6y = -20$

Multiply the first equation by 2 and the second one by 3:

$$\begin{array}{lrcll} & 6x+10y & = & 24 & \\ & 6x-18y & = & \text{-60} & \\ \text{Subtract:} & 28y & = & 84 & \Rightarrow y=3 \\ \text{Substitute back in the } 2^{nd} \text{ equation:} & 2x-18 & = & -20 & \Rightarrow x=-1 \end{array}$$

(ii) Solve the equations $x + y = 3$ and $x^2 + 2y^2 = 18$

Here the first equation is linear but the second is quadratic.
Use the linear equation for a substitution:

$$\begin{aligned} x &= 3-y \\ \Rightarrow (3-y)^2 + 2y^2 &= 18 \\ 9 - 6y + y^2 + 2y^2 &= 18 \\ 3y^2 - 6y - 9 &= 0 \\ y^2 - 2y - 3 &= 0 \end{aligned}$$

Solving this quadratic equation gives two solutions for y:

$$y = 3 \text{ or } y = -1$$

Now find the corresponding values of x using the linear equation: when $y = 3, x = 0$ and when $y = -1, x = 4$. So there are two solutions:

$$x = 0, y = 3 \quad \text{and} \quad x = 4, y = -1$$

(iii) Solve the equations $x + y + z = 6$, $y = 2x$, and $2y + z = 7$

Here we have three equations, and three variables. We use the same methods, to eliminate first one variable, then another.
Use the second equation to eliminate y from both of the others:

$$\begin{aligned} x + 2x + z = 6 \Rightarrow 3x + z &= 6 \\ 4x + z &= 7 \end{aligned}$$

Eliminate z by subtracting: $x = 1$
Work out z from $4x + z = 7$: $z = 3$
Work out y from $y = 2x$: $y = 2$
The solution is $x = 1, y = 2, z = 3$.

Exercises 1.12: Solve the following sets of simultaneous equations:

(1) $2x = 1 - y$ and $3x + 4y + 6 = 0$

(2) $2z + 3t = -0.5$ and $2t - 3z = 10.5$

(3) $x + y = a$ and $x = 2y$ for x and y, in terms of the parameter a.

(4) $a = 2b$, $a + b + c = 12$ and $2b - c = 13$

(5) $x - y = 2$ and $x^2 = 4 - 3y^2$

Further reading and exercises

- *Jacques* §1.2 covers *Simultaneous Linear Equations* thoroughly.

5. Inequalities and Absolute Value

5.1. Inequalities

$$2x + 1 \leq 6$$

is an example of an inequality. Solving the inequality means "finding the set of values of x that make the inequality true." This can be done very similarly to solving an equation:

$$\begin{aligned} 2x + 1 &\leq 6 \\ 2x &\leq 5 \\ x &\leq 2.5 \end{aligned}$$

Thus, all values of x less than or equal to 2.5 satisfy the inequality.

> When manipulating inequalities you can add anything to both sides, or subtract anything, and you can multiply or divide both sides by a positive number. But if you multiply or divide both sides by a negative number you must reverse the inequality sign.

To see why you have to reverse the inequality sign, think about the inequality:

	5	$<$	8	(which is true)
If you multiply both sides by 2, you get:	10	$<$	16	(also true)
But if you just multiplied both sides by -2, you would get:	-10	$<$	-16	(NOT true)
Instead we reverse the sign when multiplying by -2, to obtain:	-10	$>$	-16	(true)

EXAMPLES 5.1: Solve the following inequalities:

(i) $3(x + 2) > x - 4$

$$\begin{aligned} 3x + 6 &> x - 4 \\ 2x &> -10 \\ x &> -5 \end{aligned}$$

(ii) $1 - 5y \leq -9$

$$\begin{aligned} -5y &\leq -10 \\ y &\geq 2 \end{aligned}$$

5.2. Absolute Value

The absolute value, or *modulus*, of x is the positive number which has the same "magnitude" as x. It is denoted by $|x|$. For example, if $x = -6$, $|x| = 6$ and if $y = 7$, $|y| = 7$.

> $|x| = x$ if $x \geq 0$
> $|x| = -x$ if $x < 0$

EXAMPLES 5.2: *Solving equations and inequalities involving absolute values*

(i) Find the values of x satisfying $|x + 3| = 5$.

$$|x + 3| = 5 \Rightarrow x + 3 = \pm 5$$

$$\begin{aligned} \text{Either:} \quad x+3 &= 5 \quad \Rightarrow x = 2 \\ \text{or:} \quad x+3 &= -5 \quad \Rightarrow x = -8 \end{aligned}$$

So there are two solutions: $x = 2$ and $x = -8$

(ii) Find the values of y for which $|y| \leq 6$.

$$\begin{aligned} \text{Either:} \quad y &\leq 6 \\ \text{or:} \quad -y &\leq 6 \quad \Rightarrow \quad y \geq -6 \end{aligned}$$

So the solution is: $-6 \leq y \leq 6$

(iii) Find the values of z for which $|z-2| > 4$.

$$\begin{aligned} \text{Either:} \quad z-2 &> 4 \quad \Rightarrow z > 6 \\ \text{or:} \quad -(z-2) &> 4 \quad \Rightarrow z-2 < -4 \quad \Rightarrow z < -2 \end{aligned}$$

So the solution is: $z < -2$ or $z > 6$

5.3. Quadratic Inequalities

EXAMPLES 5.3: Solve the inequalities:

(i) $x^2 - 2x - 15 \leq 0$
Factorise:

$$(x-5)(x+3) \leq 0$$

If the product of two factors is negative, one must be negative and the other positive:

either: $x - 5 \leq 0$ and $x + 3 \geq 0$ $\Rightarrow -3 \leq x \leq 5$
or: $x - 5 \geq 0$ and $x + 3 \leq 0$ which is impossible.

So the solution is: $-3 \leq x \leq 5$

(ii) $x^2 - 7x + 6 > 0$

$$\Rightarrow (x-6)(x-1) > 0$$

If the product of two factors is positive, both must be positive, or both negative:

either: $x - 6 > 0$ and $x - 1 > 0$ which is true if: $x > 6$
or: $x - 6 < 0$ and $x - 1 < 0$ which is true if: $x < 1$

So the solution is: $x < 1$ or $x > 6$

EXERCISES 1.13: Solve the following equations and inequalities:

(1) (a) $2x + 1 \geq 7$ (b) $5(3 - y) < 2y + 3$

(2) (a) $|9 - 2x| = 11$ (b) $|1 - 2z| > 2$

(3) $|x + a| < 2$ where a is a parameter, and we know that $0 < a < 2$.

(4) (a) $x^2 - 8x + 12 < 0$ (b) $5x - 2x^2 \leq -3$

Further reading and exercises

- *Jacques* §1.4.1 has a little more on *Inequalities*.
- Refer to an A-level pure maths textbook for more detail and practice.

Solutions to Exercises in Chapter 1

EXERCISES 1.1:

(1) (a) 29
(b) -8
(c) -15
(d) $\frac{1}{2}$
(e) 64
(f) 3

EXERCISES 1.2:

(1) (a) $x^3 + 13x - 25$
(b) $2x^2 - 8y$ or $2(x^2 - 4y)$
(2) (a) $3z^2x + 2z - 1$
(b) $7x + 14$ or $7(x + 2)$
(3) (a) $\frac{xy}{2}$
(b) $\frac{6y}{x}$
(4) (a) $\frac{x}{4y}$
(b) $20x^3y^4$
(5) (a) y
(b) $\frac{4x^2}{y^3}$
(6) (a) $\frac{10x+3}{12}$
(b) $\frac{2}{x^2-1}$

EXERCISES 1.3:

(1) (a) $3x(1 + 2y)$
(b) $y(2y + 7)$
(c) $3(2a + b + 3c)$
(2) (a) $x^2(3x - 10)$
(b) $c(a - b)$
(3) $(x + 2)(y + 2z)$
(4) $x(2 - x)$

EXERCISES 1.4:

(1) $(x + 1)(x + 3)$
(2) $(y - 5)(y - 2)$
(3) $(2x + 1)(x + 3)$
(4) $(z + 5)(z - 3)$
(5) $(2x + 3)(2x - 3)$
(6) $(y - 5)^2$
(7) Not possible to split into integer factors.

EXERCISES 1.5:

(1) (a) $= \sqrt{2 \times 18} = \sqrt{36} = 6$
(b) $= \sqrt{49 \times 5} = 7\sqrt{5}$
(c) $\frac{15}{\sqrt{3}} = \frac{15\sqrt{3}}{3} = 5\sqrt{3}$
(2) (a) $\sqrt{5}$
(b) $4x^2$
(c) $\frac{x}{2}$
(d) $\sqrt{2}y$

EXERCISES 1.6:

(1) (a) 16
(b) 6
(c) 3
(d) $\frac{1}{4}$
(2) (a) $2x^{11}$
(b) $\frac{1}{x}$
(c) $\log_{10} y$
(d) 3
(3) (a) $(9ab)^3$
(b) $-\frac{5}{3}\log_{10} b$

EXERCISES 1.7:

(1) $x = 3$
(2) $y = -3$
(3) $x = 2$
(4) $z = -1$
(5) $a = -\frac{1}{3}$

EXERCISES 1.8:

(1) $x = \frac{6}{a}$
(2) $y = -4b$
(3) $z = \frac{a+b}{2}$

EXERCISES 1.9:

(1) $t = \frac{v-u}{a}$
(2) $a = \sqrt{c^2 - b^2}$
(3) $p = 1200r - 6n$
(4) $n = \frac{W}{m^2} + 3$

EXERCISES 1.10:

(1) $x = \frac{-3\pm\sqrt{61}}{2}$
(2) $y = 1.5$
(3) $z = -\frac{4}{3}, 2$
(4) $x = \frac{7\pm\sqrt{33}}{4}$
(5) No solutions.
(6) $x = \frac{7\pm\sqrt{17}}{4}$
(7) $x = 3k$
(8) $y = (m \pm \sqrt{m^2 - 1})$ No solution if $-1 < m < 1$

EXERCISES 1.11:

(1) $x = 14$
(2) $t = \frac{\log(16)}{3\log(5)} = 0.5742$
(3) $x = 6^{\frac{1}{0.4}} = 88.18$
(4) $x = (0.74)^{\frac{1}{0.42}} = 0.4883$
(5) $x = \pm 3$
(6) $y = \pm 2$
(7) $n = \frac{\log_2 3}{\log_2 \frac{2}{3}} = -2.7095$
(8) $x = 4, x = 1$

EXERCISES 1.12:

(1) $x = 2$, $y = -3$
(2) $t = 1.5$, $z = -2.5$
(3) $x = \frac{2a}{3}$, $y = \frac{a}{3}$
(4) $a = 10$, $b = 5$, $c = -3$
(5) $(x, y) = (2, 0)$
$(x, y) = (1, -1)$

EXERCISES 1.13:

(1) (a) $x \geq 3$
(b) $\frac{12}{7} < y$
(2) (a) $x = -1, 10$
(b) $z < -0.5$ or $z > 1.5$
(3) $-2 - a < x < 2 - a$
(4) (a) $2 < x < 6$
(b) $x \geq 3$, $x \leq -0.5$

[2]This Version of Workbook Chapter 1: September 25, 2014

Worksheet 1: Review of Algebra

(1) For a firm, the cost of producing q units of output is $C = 4 + 2q + 0.5q^2$. What is the cost of producing (a) 4 units (b) 1 unit (c) no units?

(2) Evaluate the expression $x^3(y + 7)$ when $x = -2$ and $y = -10$.

(3) Simplify the following algebraic expressions, factorising the answer where possible:
(a) $x(2y+3x-12)-3(2-5xy)-(3x+8xy-6)$ (b) $z(2-3z+5z^2)+3(z^2-z^3-4)$

(4) Simplify: (a) $6a^4b \times 4b \div 8ab^3c$ (b) $\sqrt{3x^3y} \div \sqrt{27xy}$ (c) $(2x^3)^3 \times (xz^2)^4$

(5) Write as a single fraction: (a) $\dfrac{2y}{3x} + \dfrac{4y}{5x}$ (b) $\dfrac{x+1}{4} - \dfrac{2x-1}{3}$

(6) Factorise the following quadratic expressions:
(a) $x^2 - 7x + 12$ (b) $16y^2 - 25$ (c) $3z^2 - 10z - 8$

(7) Evaluate (without using a calculator): (a) $4^{\frac{3}{2}}$ (b) $\log_{10} 100$ (c) $\log_5 125$

(8) Write as a single logarithm: (a) $2\log_a(3x) + \log_a x^2$ (b) $\log_a y - 3\log_a z$

(9) Solve the following equations:
(a) $5(2x - 9) = 2(5 - 3x)$ (b) $1 + \dfrac{6}{y-8} = -1$ (c) $z^{0.4} = 7$ (d) $3^{2t-1} = 4$

(10) Solve these equations for x, in terms of the parameter a:
(a) $ax - 7a = 1$ (b) $5x - a = \dfrac{x}{a}$ (c) $\log_a(2x + 5) = 2$

(11) Make Q the subject of: $P = \sqrt{\dfrac{a}{Q^2 + b}}$

(12) Solve the equations: (a) $7 - 2x^2 = 5x$ (b) $y^2 + 3y - 0.5 = 0$ (c) $|1 - z| = 5$

(13) Solve the simultaneous equations:
(a) $2x - y = 4$ and $5x = 4y + 13$
(b) $y = x^2 + 1$ and $2y = 3x + 4$

(14) Solve the inequalities: (a) $2y - 7 \leq 3$ (b) $3 - z > 4 + 2z$ (c) $3x^2 < 5x + 2$

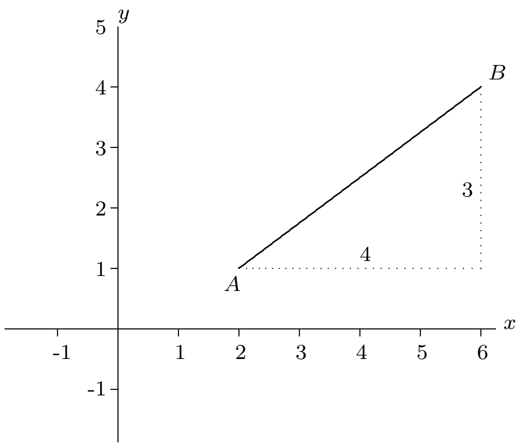
CHAPTER 2

Lines and Graphs

Almost everything in this chapter is revision from GCSE maths. It reminds you how to draw graphs, and focuses in particular on **straight line graphs** and their **gradients**. We also look at graphs of quadratic functions, and use graphs to solve equations and inequalities. An important economic application of straight line graphs is **budget constraints**.

1. The Gradient of a Line

A is a point with co-ordinates $(2,1)$; B has co-ordinates $(6,4)$.



When you move from A to B, the change in the x-coordinate is

$$\Delta x = 6 - 2 = 4$$

and the change in the y-coordinate is

$$\Delta y = 4 - 1 = 3$$

The gradient (or slope) of AB is Δy divided by Δx:

$$\text{Gradient} = \frac{\Delta y}{\Delta x} = \frac{3}{4} = 0.75$$

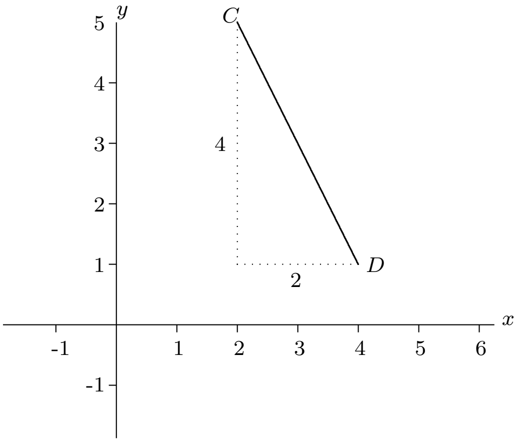
(The symbol Δ, pronounced "delta", denotes "change in".)

It doesn't matter which end of the line you start. If you move from B to A, the changes are negative, but the gradient is the same: $\Delta x = 2 - 6 = -4$ and $\Delta y = 1 - 4 = -3$, so the gradient is $(-3)/(-4) = 0.75$.

There is a general formula:

The gradient of the line joining (x_1, y_1) and (x_2, y_2) is:

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



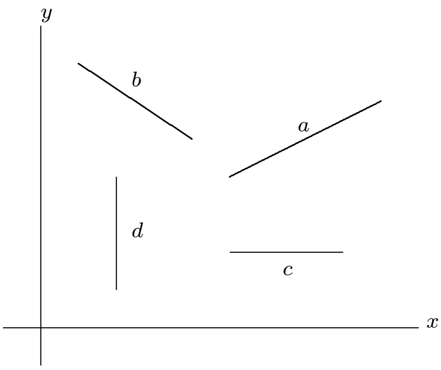
Here the gradient is negative. When you move from C to D:

$$\Delta x = 4 - 2 = 2$$

$$\Delta y = 1 - 5 = -4$$

The gradient of CD is:

$$\frac{\Delta y}{\Delta x} = \frac{-4}{2} = -2$$



In this diagram the gradient of line a is positive, and the gradient of b is negative: as you move in the x-direction, a goes uphill, but b goes downhill.

The gradient of c is zero. As you move along the line the change in the y-coordinate is zero: $\Delta y = 0$

The gradient of d is infinite. As you move along the line the change in the x-coordinate is zero (so if you tried to calculate the gradient you would be dividing by zero).

Exercises 2.1: **Gradients**

(1) Plot the points $A(1, 2), B(7, 10), C(-4, 14), D(9, 2)$ and $E(-4, -1)$ on a diagram.

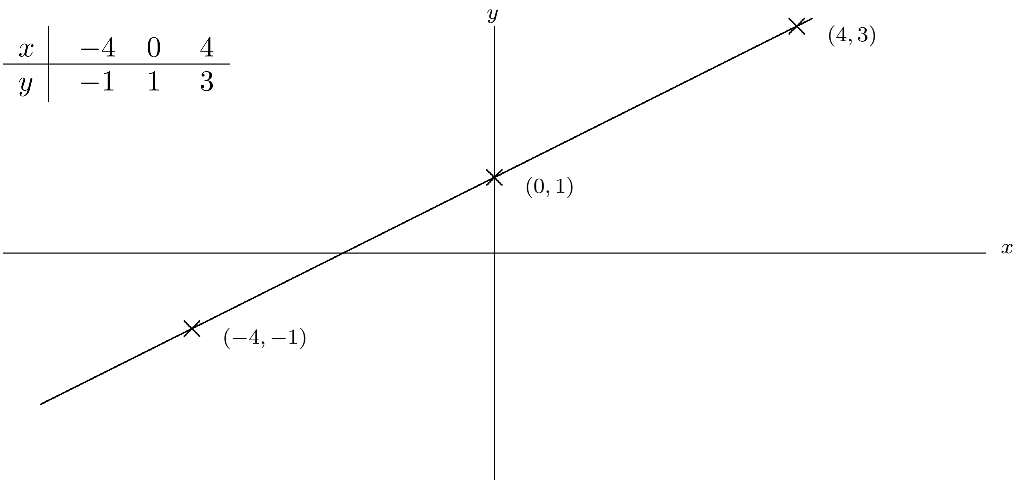
(2) Find the gradients of the lines AB, AC, CE, AD.

2. Drawing Graphs

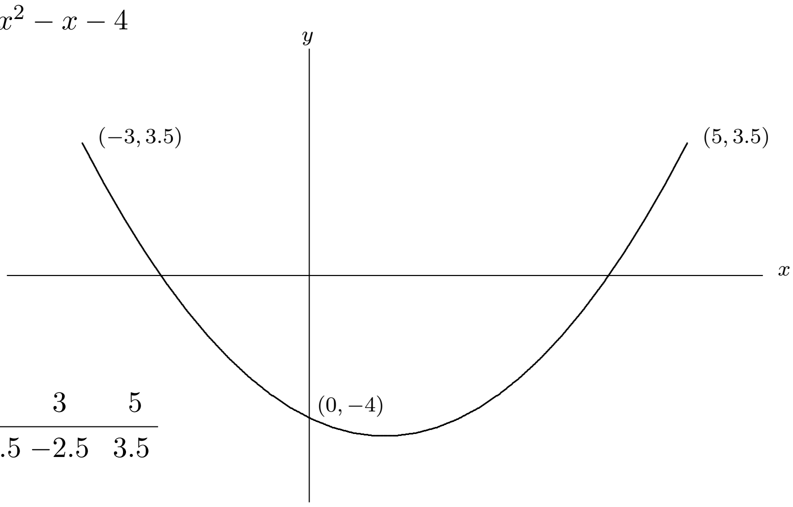
The equation $y = 0.5x+1$ expresses a relationship between 2 quantities x and y (or a formula for y in terms of x) that can be represented as a graph in x-y space. To draw the graph, calculate y for a range of values of x, then plot the points and join them with a curve or line.

Examples 2.1: $y = 0.5x + 1$

x	-4	0	4
y	-1	1	3



Examples 2.2: $y = 0.5x^2 - x - 4$



x	-3	-1	0	1	3	5
y	3.5	-2.5	-4	-4.5	-2.5	3.5

Exercises 2.2: Draw the graphs of the following relationships:

(1) $y = 3x - 2$ for values of x between -4 and $+4$.

(2) $P = 10-2Q$, for values of Q between 0 and 5. (This represents a demand function: the relationship between the market price P and the total quantity sold Q.)

(3) $y = 4/x$, for values of x between -4 and $+4$.

(4) $C = 3+2q^2$, for values of q between 0 and 4. (This represents a firm's cost function: its total costs are C if it produces a quantity q of goods.)

3. Straight Line (Linear) Graphs

EXERCISES 2.3: **Straight Line Graphs**

(1) Using a diagram with x and y axes from -4 to $+4$, draw the graphs of:

(a) $y = 2x$
(b) $2x + 3y = 6$
(c) $y = 1 - 0.5x$
(d) $y = -3$
(e) $x = 4$

(2) For each graph find (i) the gradient, and (ii) the *vertical intercept* (that is, the value of y where the line crosses the y-axis, also known as the y-intercept).

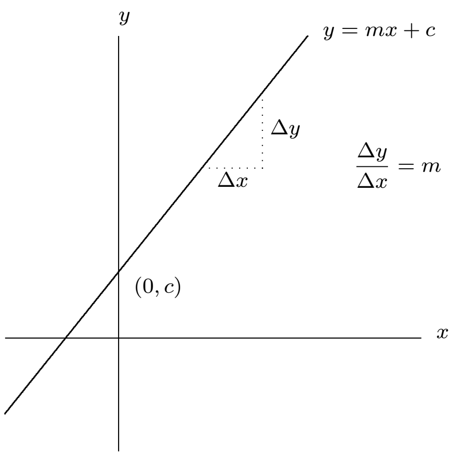
Each of the first four equations in this exercise can be rearranged to have the form $y = mx+c$:

(a) $y = 2x$	$\Rightarrow y = 2x + 0$	$\Rightarrow m = 2$	$c = 0$
(b) $2x + 3y = 6$	$\Rightarrow y = -\frac{2}{3}x + 2$	$\Rightarrow m = -\frac{2}{3}$	$c = 2$
(c) $y = 1 - 0.5x$	$\Rightarrow y = -0.5x + 1$	$\Rightarrow m = -0.5$	$c = 1$
(d) $y = -3$	$\Rightarrow y = 0x - 3$	$\Rightarrow m = 0$	$c = -3$

(e) is a special case. It cannot be written in the form $y = mx + c$, its gradient is infinite, and it has no vertical intercept.

Check these values of m and c against your answers. You should find that m is the gradient and c is the y-intercept.

Note that in an equation of the form $y = mx + c$, y is equal to a polynomial of degree 1 in x (see Chapter 1).



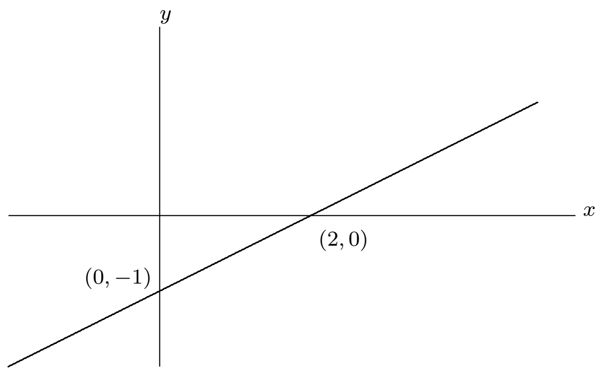
If an equation can be written in the form $y = mx + c$, then the graph is a straight line, with gradient m and vertical intercept c. We say "y is a linear function of x."

Examples 3.1: Sketch the line $x - 2y = 2$

"Sketching" a graph means drawing a picture to indicate its general shape and position, rather than plotting it accurately. First rearrange the equation:

$$y = 0.5x - 1$$

So the gradient is 0.5 and the y-intercept is -1. We can use this to sketch the graph.

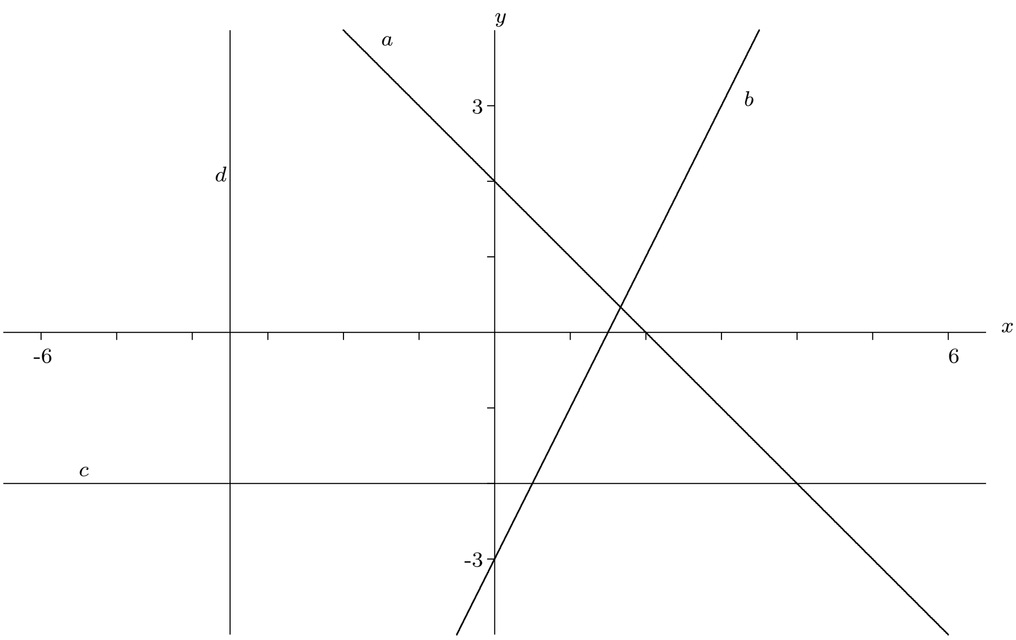


Exercises 2.4: $y = mx + c$

(1) For each of the lines in the diagram below, work out the gradient and hence write down the equation of the line.

(2) By writing each of the following lines in the form $y = mx + c$, find its gradient:
(a) $y = 4 - 3x$ (b) $3x + 5y = 8$ (c) $x + 5 = 2y$ (d) $y = 7$ (e) $2x = 7y$

(3) By finding the gradient and y-intercept, sketch each of the following straight lines:
$y = 3x + 5$
$y + x = 6$
$3y + 9x = 8$
$x = 4y + 3$

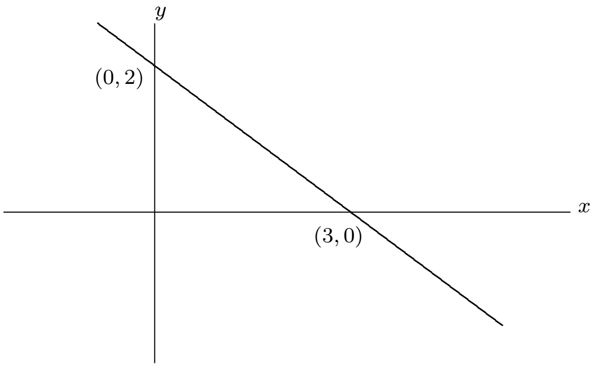


3.1. Lines of the Form $ax + by = c$

Lines such as $2x+3y = 6$ can be rearranged to have the form $y = mx+c$, and hence sketched, as in the previous exercise. But it is easier in this case to work out what the line is like by finding the points where it crosses both axes.

EXAMPLES 3.2:
Sketch the line $2x + 3y = 6$.

When $x = 0, y = 2$
When $y = 0, x = 3$



> EXERCISES 2.5: Sketch the following lines: (1) $4x + 5y = 100$ (2) $2y + 6x = 7$

3.2. Working out the Equation of a Line

EXAMPLES 3.3: What is the equation of the line

(i) with gradient 3, passing through $(2, 1)$?
gradient $= 3$ $\Rightarrow y = 3x + c$
$y = 1$ when $x = 2$ $\Rightarrow 1 = 6 + c$ $\Rightarrow c = -5$
The line is $y = 3x - 5$.

(ii) passing through $(-1, -1)$ and $(5, 14)$?
First work out the gradient: $\frac{14-(-1)}{5-(-1)} = \frac{15}{6} = 2.5$
gradient $= 2.5$ $\Rightarrow y = 2.5x + c$
$y = 14$ when $x = 5$ $\Rightarrow 14 = 12.5 + c \Rightarrow c = 1.5$
The line is $y = 2.5x + 1.5$ (or equivalently $2y = 5x + 3$).

There is a formula that you can use (although the method above is just as good):

> The equation of a line with gradient m, passing through the point (x_1, y_1) is: $y = m(x - x_1) + y_1$

> EXERCISES 2.6: Find the equations of the following lines:
> (1) passing through $(4, 2)$ with gradient 7
> (2) passing through $(0, 0)$ with gradient 1
> (3) passing through $(-1, 0)$ with gradient -3
> (4) passing through $(-3, 4)$ and parallel to the line $y + 2x = 5$
> (5) passing through $(0, 0)$ and $(5, 10)$
> (6) passing through $(2, 0)$ and $(8, -1)$

4. Quadratic Graphs

If we can write a relationship between x and y so that y is equal to a quadratic polynomial in x (see Chapter 1):

$$y = ax^2 + bx + c$$

where a, b and c are numbers, then we say "y is a quadratic function of x", and the graph is a parabola (a U-shape) like the one in Example 2.2.

EXERCISES 2.7: **Quadratic Graphs**

(1) Draw the graphs of (i) $y = 2x^2 - 5$ (ii) $y = -x^2 + 2x$, for values of x between -3 and $+3$.

(2) For each graph note that: if a (the coefficient of x^2) is positive, the graph is a U-shape; if a is negative then it is an inverted U-shape; the vertical intercept is given by c; and the graph is symmetric.

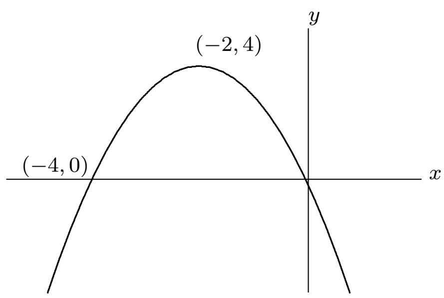
So, to sketch the graph of a quadratic you can:
- decide whether it is a U-shape or an inverted U;
- find the y-intercept;
- find the points where it crosses the x-axis (if any), by solving $ax^2 + bx + c = 0$;
- find its maximum or minimum point using symmetry: find two points with the same y-value, then the max or min is at the x-value halfway between them.

EXAMPLES 4.1: Sketch the graph of $y = -x^2 - 4x$

- $a = -1$, so it is an inverted U-shape.
- The y-intercept is 0.
- Solving $-x^2 - 4x = 0$ to find where it crosses the x-axis:

$$x^2 + 4x = 0 \quad \Rightarrow \quad x(x+4) = 0$$
$$\Rightarrow \quad x = 0 \text{ or } x = -4$$

- Its maximum point is halfway between these two points, at $x = -2$, and at this point $y = -4 - 4(-2) = 4$.



EXERCISES 2.8: Sketch the graphs of the following quadratic functions:

(1) $y = 2x^2 - 18$

(2) $y = 4x - x^2 + 5$

5. Solving Equations and Inequalities using Graphs

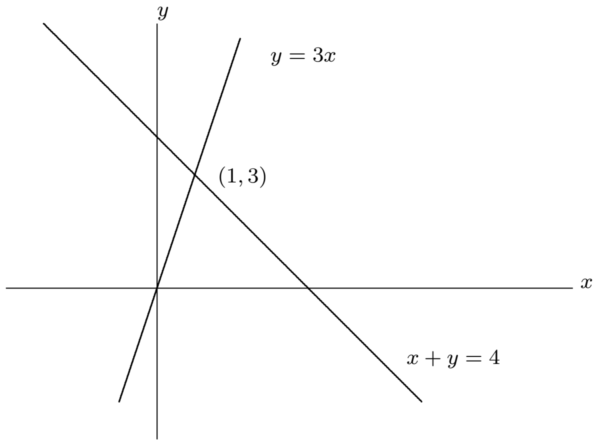
5.1. Solving Simultaneous Equations

The equations:

$$x + y = 4$$
$$y = 3x$$

could be solved algebraically (see Chapter 1). Alternatively we could draw their graphs, and find the point where they intersect.

The solution is $x = 1, y = 3$.



5.2. Solving Quadratic Equations

We could solve the solve the quadratic equation $x^2 - 5x + 2 = 0$ using the quadratic formula (see Chapter 1). Alternatively we could find an approximate solution by drawing the graph of $y = x^2 - 5x + 2$ (as accurately as possible), and finding where it crosses the x-axis (that is, finding the points where $y = 0$).

EXERCISES 2.9: **Solving Equations using Graphs**

(1) Solve the simultaneous equations $y = 4x - 7$ and $y = x - 1$ by drawing (accurately) the graphs of the two lines.

(2) By sketching their graphs, explain why you cannot solve either of the following pairs of simultaneous equations:

$$y = 3x - 5 \quad \text{and} \quad 2y - 6x = 7;$$
$$x - 5y = 4 \quad \text{and} \quad y = 0.2x - 0.8$$

(3) By sketching their graphs, show that the simultaneous equations $y = x^2$ and $y = 3x + 4$ have two solutions. Find the solutions algebraically.

(4) Show algebraically that the simultaneous equations $y = x^2 + 1$ and $y = 2x$ have only one solution. Draw the graph of $y = x^2 + 1$ and use it to show that the simultaneous equations:

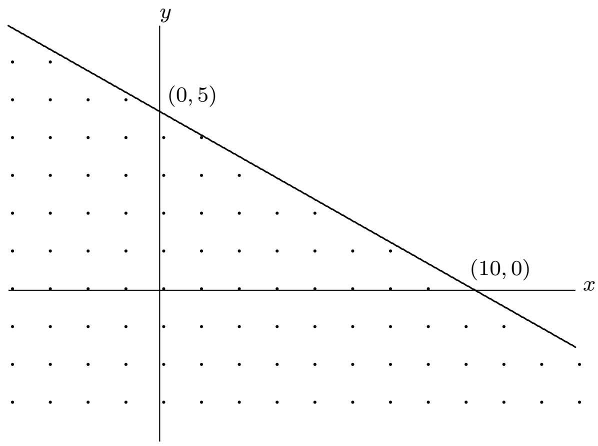
$$y = x^2 + 1 \text{ and } y = mx$$

have: no solutions if $0 < m < 2$; one solution if $m = 2$; and two solutions if $m > 2$.

(5) Sketch the graph of the quadratic function $y = x^2 - 8x$. From your sketch, find the approximate solutions to the equation $x^2 - 8x = -4$.

5.3. Representing Inequalities using Graphs

If we draw the graph of the line $2y+x=10$, all the points satisfying $2y+x<10$ lie on one side of the line. The dotted region shows the inequality $2y+x<10$.



> EXERCISES 2.10: **Representing Inequalities**
> (1) Draw a sketch showing all the points where $x+y<1$ and $y<x+1$ and $y>-3$.
> (2) Sketch the graph of $y=3-2x^2$, and show the region where $y<3-2x^2$.

5.4. Using Graphs to Help Solve Quadratic Inequalities

In Chapter 1, we solved quadratic inequalities such as: $x^2-2x-15\le 0$. To help do this quickly, you can sketch the graph of the quadratic polynomial $y=x^2-2x-15$.

EXAMPLES 5.1: *Quadratic Inequalities*

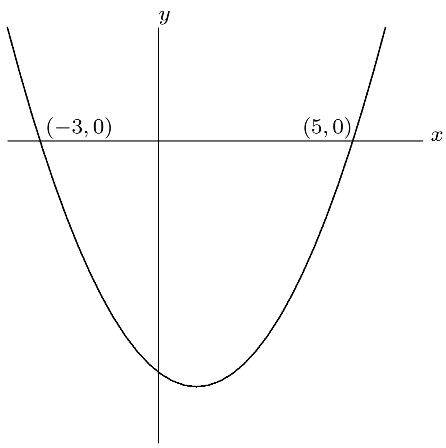
(i) Solve the inequality $x^2-2x-15\le 0$.
As before, the first step is to factorise:

$$(x-5)(x+3)\le 0$$

Now sketch the graph of $y=x^2-2x-15$.
We can see from the graph that $x^2-2x-15\le 0$ when:

$$-3\le x\le 5$$

This is the solution of the inequality.



(ii) Solve the inequality $x^2-2x-15>0$.
From the same graph the solution is: $x>5$ or $x<-3$.

> EXERCISES 2.11: Solve the inequalities:
> (1) $x^2-5x>0$
> (2) $3x+5-2x^2\ge 0$
> (3) $x^2-3x+1\le 0$ (Hint: you will need to use the quadratic formula for the last one.)

6. Economic Application: Budget Constraints

6.1. An Example

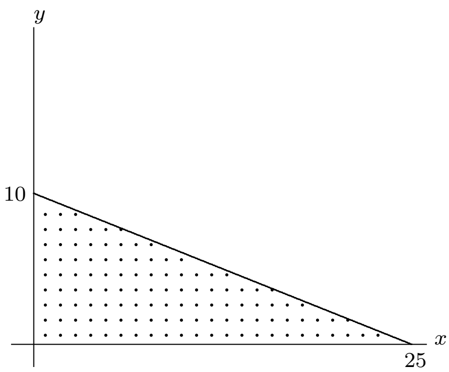
Suppose pencils cost 20p, and pens cost 50p. If a student buys x pencils and y pens, the total amount spent is:

$$20x + 50y$$

If the maximum amount he has to spend on writing implements is £5 his budget constraint is:

$$20x + 50y \leq 500$$

His budget set (the choices of pens and pencils that he can afford) can be shown as the shaded area on a diagram:



From the equation of the budget line:

$$20x + 50y = 500$$

we can see that the gradient of the budget line is:

$$-\frac{20}{50}$$

(You could rewrite the equation in the form $y = mx + c$.)

6.2. The General Case

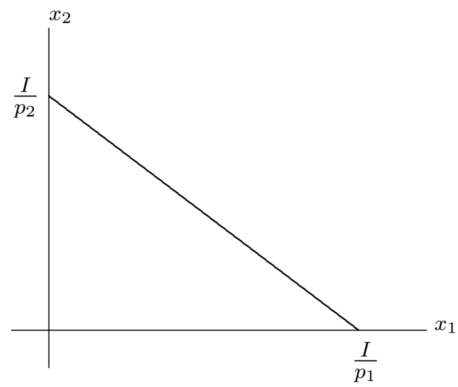
Suppose that a consumer has a choice of two goods, good 1 and good 2. The price of good 1 is p_1 and the price of good 2 is p_2. If he buys x_1 units of good 1, and x_2 units of good 2, the total amount spent is:

$$p_1x_1 + p_2x_2$$

If the maximum amount he has to spend is his income I his budget constraint is:

$$p_1x_1 + p_2x_2 \leq I$$

To draw the budget line:



$$p_1x_1 + p_2x_2 = I$$

note that it crosses the x_1-axis where:

$$x_2 = 0 \Rightarrow p_1x_1 = I \Rightarrow x_1 = \frac{I}{p_1}$$

and similarly for the x_2 axis.

The gradient of the budget line is:

$$-\frac{p_1}{p_2}$$

EXERCISES 2.12: **Budget Constraints**

(1) Suppose that the price of coffee is 30p and the price of tea is 25p. If a consumer has a daily budget of £1.50 for drinks, draw his budget set and find the slope of the budget line. What happens to his budget set and the slope of the budget line if his drinks budget increases to £2?

(2) A consumer has budget constraint $p_1x_1 + p_2x_2 \leq I$. Show diagrammatically what happens to the budget set if

(a) income I increases

(b) the price of good 1, p_1, increases

(c) the price of good 2, p_2, decreases.

Further reading and exercises

- *Jacques* §1.1 has more on co-ordinates and straight-line graphs.
- *Jacques* §2.1 includes graphs of quadratic functions.
- *Varian* discusses budget constraints in detail.

Solutions to Exercises in Chapter 2

Exercises 2.1:

(1)

(2) gradient (AB) $= \frac{4}{3}$,
(AC) $= -\frac{12}{5}$,
(CE) $= \infty$,
(AD) $= 0$

Exercises 2.2:

Exercises 2.3:

(1)

(2) (a) (i) 2 (ii) $y = 0$
(b) (i) $-\frac{2}{3}$ (ii) $y = 2$
(c) (i) $-\frac{1}{2}$ (ii) $y = 1$
(d) (i) 0 (ii) $y = -3$
(e) (i) ∞ (ii) None.

Exercises 2.4:

(1) (a) $y = 2 - x$
(b) $y = 2x - 3$
(c) $y = -2$
(d) $x = -3\frac{1}{2}$

(2) (a) -3
(b) $-\frac{3}{5}$
(c) $\frac{1}{2}$
(d) 0
(e) $\frac{2}{7}$

Exercises 2.5:

Exercises 2.6:

(1) $y = 7x - 26$
(2) $y = x$
(3) $y + 3x = -3$
(4) $y + 2x = -2$
(5) $y = 2x$
(6) $6y + x = 2$

Exercises 2.7:

Exercises 2.8:

Exercises 2.9:

(1) $(x, y) = (2, 1)$
(2) The lines are parallel.
(3) $(x, y) = (-1, 1)$ and $(4, 16)$
(4) $(x, y) = (1, 2)$
(5) $(x, y) \approx (0.536, -4)$ and $(7.464, -4)$ Actual values for $x = 4 \pm \sqrt{12}$

Exercises 2.10:

Exercises 2.11:

(1) $x > 5$ or $x < 0$
(2) $-1 \leq x \leq \frac{5}{2}$
(3) $\frac{3-\sqrt{5}}{2} \leq x \leq \frac{3+\sqrt{5}}{2}$

Exercises 2.12:

(1) 6 Coffee + 5 Tea $\leq$ 30. Gradient (with Coffee on horiz. axis) $= -\frac{6}{5}$
Budget line shifts outwards; budget set larger; slope the same.

(2)
(a) Budget line shifts outwards; budget set larger; slope the same.
(b) Budget line pivots around intercept with x_2-axis; budget set smaller; (absolute value of) gradient increases.
(c) Budget line pivots around intercept with x_1-axis; budget set larger; (absolute value of) gradient increases.

[1]This Version of Workbook Chapter 2: August 22, 2003

Worksheet 2: Lines and Graphs

(1) Find the gradients of the lines AB, BC, and CA where A is the point $(5, 7)$, B is $(-4, 1)$ and C is $(5, -17)$.

(2) Draw (accurately), for values of x between -5 and 5, the graphs of:
(a) $y - 2.5x = -5$ (b) $z = \frac{1}{4}x^2 + \frac{1}{2}x - 1$
Use (b) to solve the equation $\frac{1}{4}x^2 + \frac{1}{2}x = 1$

(3) Find the gradient and y-intercept of the following lines:
(a) $3y = 7x - 2$ (b) $2x + 3y = 12$ (c) $y = -x$

(4) Sketch the graphs of $2P = Q+5$, $3Q+4P = 12$, and $P = 4$, with Q on the horizontal axis.

(5) What is the equation of the line through $(1, 1)$ and $(4, -5)$?

(6) Sketch the graph of $y = 3x - x^2 + 4$, and hence solve the inequality $3x - x^2 < -4$.

(7) Draw a diagram to represent the inequality $3x - 2y < 6$.

(8) Electricity costs 8p per unit during the daytime and 2p per unit if used at night. The quarterly charge is £10. A consumer has £50 to spend on electricity for the quarter.
   (a) What is his budget constraint?
   (b) Draw his budget set (with daytime units as "good 1" on the horizontal axis).
   (c) Is the bundle (440, 250) in his budget set?
   (d) What is the gradient of the budget line?

(9) A consumer has a choice of two goods, good 1 and good 2. The price of good 2 is 1, and the price of good 1 is p. The consumer has income M.
   (a) What is the budget constraint?
   (b) Sketch the budget set, with good 1 on the horizontal axis, assuming that $p > 1$.
   (c) What is the gradient of the budget line?
   (d) If the consumer decides to spend all his income, and buy equal amounts of the two goods, how much of each will he buy?
   (e) Show on your diagram what happens to the budget set if the price of good 1 falls by 50%.

CHAPTER 3

Sequences, Series, and Limits; the Economics of Finance

If you have done A-level maths you will have studied Sequences and Series (in particular **Arithmetic** and **Geometric** ones) before; if not you will need to work carefully through the first two sections of this chapter. Sequences and series arise in many economic applications, such as the **economics of finance and investment**. Also, they help you to understand the concept of a **limit** and the significance of **the natural number**, e. You will need both of these later.

1. Sequences and Series

1.1. Sequences

A sequence is a set of terms (or numbers) arranged in a definite order.

Examples 1.1: *Sequences*

(i) $3, 7, 11, 15, \ldots$
In this sequence each term is obtained by adding 4 to the previous term. So the next term would be 19.

(ii) $4, 9, 16, 25, \ldots$
This sequence can be rewritten as $2^2, 3^2, 4^2, 5^2, \ldots$ The next term is 6^2, or 36.

The dots($\ldots$) indicate that the sequence continues indefinitely – it is an *infinite sequence*. A sequence such as $3, 6, 9, 12$ (stopping after a finite number of terms) is a *finite sequence*. Suppose we write u_1 for the first term of a sequence, u_2 for the second and so on. There may be a formula for u_n, the n^{th} term:

Examples 1.2: *The n^{th} term of a sequence*

(i) $4, 9, 16, 25, \ldots$ The formula for the n^{th} term is $u_n = (n+1)^2$.

(ii) $u_n = 2n + 3$. The sequence given by this formula is: $5, 7, 9, 11, \ldots$

(iii) $u_n = 2^n + n$. The sequence is: $3, 6, 11, 20, \ldots$

Or there may be a formula that enables you to work out the terms of a sequence from the preceding one(s), called a *recurrence relation*:

Examples 1.3: *Recurrence Relations*

(i) Suppose we know that: $u_n = u_{n-1} + 7n$ and $u_1 = 1$.
Then we can work out that $u_2 = 1 + 7 \times 2 = 15$, $u_3 = 15 + 7 \times 3 = 36$, and so on, to find the whole sequence : $1, 15, 36, 64, \ldots$

(ii) $u_n = u_{n-1} + u_{n-2}$, $u_1 = 1$, $u_2 = 1$
The sequence defined by this formula is: $1, 1, 2, 3, 5, 8, 13, \ldots$

1.2. Series

A series is formed when the terms of a sequence are added together. The Greek letter Σ (pronounced "sigma") is used to denote "the sum of":

$$\sum_{r=1}^{n} u_r \text{ means } u_1 + u_2 + \cdots + u_n$$

EXAMPLES 1.4: *Series*

(i) In the sequence $3, 6, 9, 12, \ldots$, the sum of the first five terms is the series: $3 + 6 + 9 + 12 + 15$.

(ii) $\displaystyle\sum_{r=1}^{6}(2r+3) = 5 + 7 + 9 + 11 + 13 + 15$

(iii) $\displaystyle\sum_{r=5}^{k}\frac{1}{r^2} = \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \cdots + \frac{1}{k^2}$

EXERCISES 3.1: **Sequences and Series**

(1) Find the next term in each of the following sequences:

(a) $2, 5, 8, 11, \ldots$
(b) $0.25, 0.75, 1.25, 1.75, 2.25, \ldots$
(c) $5, -1, -7, \ldots$
(d) $36, 18, 9, 4.5, \ldots$
(e) $1, -2, 3, -4, 5, \ldots$

(2) Find the 2^{nd}, 4^{th} and 6^{th} terms in the sequence given by: $u_n = n^2 - 10$

(3) If $u_n = \frac{u_{n-1}}{2} + 2$ and $u_1 = 4$ write down the first five terms of the sequence.

(4) If $u_n = u_{n-1}^2 + 3u_{n-1}$ and $u_3 = -2$, find the value of u_4.

(5) Find the value of $\sum_{r=1}^{4} 3r$

(6) Write out the following sums without using sigma notation:

(a) $\sum_{r=1}^{5}\frac{1}{r^2}$ (b) $\sum_{i=0}^{3} 2^i$ (c) $\sum_{j=0}^{n}(2j+1)$

(7) In the series $\sum_{i=0}^{n-1}(4i+1)$, (a) how many terms are there? (b) what is the formula for the last term?

(8) Express using the Σ notation:

(a) $1^2 + 2^2 + 3^2 + ... + 25^2$
(b) $6 + 9 + 12 + \cdots + 21$
(c) $16 + 25 + 36 + 49 + \cdots + n^2$

Further reading and exercises

- For more practice with simple sequences and series, you could use an A-level pure maths textbook.

2. Arithmetic and Geometric Sequences

2.1. Arithmetic Sequences

An *arithmetic sequence* is one in which each term can be obtained by adding a fixed number (called the *common difference*) to the previous term.

EXAMPLES 2.1: *Some Arithmetic Sequences*

(i) $1, 3, 5, 7, \ldots$ The common difference is 2.

(ii) $13, 7, 1, -5, \ldots$ The common difference is -6.

> In an arithmetic sequence with first term a and common difference d, the formula for the n^{th}term is:
> $u_n = a + (n-1)d$

EXAMPLES 2.2: *Arithmetic Sequences*

(i) What is the 10^{th} term of the arithmetic sequence $5, 12, 19, \ldots$?
In this sequence $a = 5$ and $d = 7$. So the 10^{th} term is: $5 + 9 \times 7 = 68$.

(ii) If an arithmetic sequence has $u_{10} = 24$ and $u_{11} = 27$, what is the first term?
The common difference, d, is 3. Using the formula for the 11th term:

$$27 = a + 10 \times 3$$

Hence the first term, a, is -3.

2.2. Arithmetic Series

When the terms in an arithmetic sequence are summed, we obtain an arithmetic series. Suppose we want to find the sum of the first 5 terms of the arithmetic sequence with first term 3 and common difference 4. We can calculate it directly:

$$S_5 = 3 + 7 + 11 + 15 + 19 = 55$$

But there is a general formula:

> If an arithmetic sequence has first term a and common difference d, the sum of the first n terms is:
> $S_n = \frac{n}{2}(2a + (n-1)d)$

We can check that the formula works: $S_5 = \frac{5}{2}(2 \times 3 + 4 \times 4) = 55$

2.3. To Prove the Formula for an Arithmetic Series

Write down the series in order and in reverse order, then add them together, pairing terms:

$$\begin{array}{rcccccccc} S_n = & a & + & (a+d) & + & (a+2d) & + \cdots + & (a+(n-1)d) \\ S_n = & (a+(n-1)d) & + & (a+(n-2)d) & + & (a+(n-3)d) & + \cdots + & a \\ 2S_n = & (2a+(n-1)d) & + & (2a+(n-1)d) & + & (2a+(n-1)d) & + \cdots + & (2a+(n-1)d) \\ = & n\,(2a+(n-1)d) & & & & & & \end{array}$$

Dividing by 2 gives the formula.

EXERCISES 3.2: **Arithmetic Sequences and Series**

(1) Using the notation above, find the values of a and d for the arithmetic sequences:
(a) $4, 7, 10, 13, \ldots$ (b) $-1, 2, 5, \ldots$ (c) $-7, -8.5, -10, -11.5, \ldots$

(2) Find the n^{th} term in the following arithmetic sequences:
(a) $44, 46, 48, \ldots$ (b) $-3, -7, -11, -15, \ldots$

(3) If an arithmetic sequence has $u_{20} = 100$, and $u_{22} = 108$, what is the first term?

(4) Use the formula for an arithmetic series to calculate the sum of the first 8 terms of the arithmetic sequence with first term 1 and common difference 10.

(5) (a) Find the values of a and d for the arithmetic sequence: $21, 19, 17, 15, 13, \ldots$
(b) Use the formula for an arithmetic series to calculate $21 + 19 + 17 + 15 + 13$.
(c) Now use the formula to calculate the sum of $21+19+17+\ldots+1$.

(6) Is the sequence $u_n = -4n + 2$ arithmetic? If so, what is the common difference?

2.4. Geometric Sequences

A *geometric sequence* is one in which each term can be obtained by multiplying the previous term by a fixed number, called the *common ratio*.

EXAMPLES 2.3: *Geometric Sequences*

(i) $\frac{1}{2}, 1, 2, 4, 8, \ldots$ Each term is double the previous one. The common ratio is 2.

(ii) $81, 27, 9, 3, 1, \ldots$ The common ratio is $\frac{1}{3}$.

In a geometric sequence with first term a and common ratio r, the formula for the n^{th} term is:
$u_n = ar^{n-1}$

EXAMPLES 2.4: Consider the geometric sequence with first term 2 and common ratio 1.1.

(i) What is the 10^{th} term?
Applying the formula, with $a = 2$ and $r = 1.1$, $u_{10} = 2 \times (1.1)^9 = 4.7159$

(ii) Which terms of the sequence are greater than 20?
The n^{th} term is given by $u_n = 2 \times (1.1)^{n-1}$. It exceeds 20 if:

$$2 \times (1.1)^{n-1} > 20$$
$$(1.1)^{n-1} > 10$$

Taking logs of both sides (see chapter 1, section 5):

$$\begin{aligned} \log_{10}(1.1)^{n-1} &> \log_{10} 10 \\ (n-1)\log_{10} 1.1 &> 1 \\ n &> \frac{1}{\log_{10} 1.1} + 1 = 25.2 \end{aligned}$$

So all terms from the 26^{th} onwards are greater than 20.

2.5. Geometric Series

Suppose we want to find the sum of the first 10 terms of the geometric sequence with first term 3 and common ratio 0.5:

$$S_{10} = 3 + 1.5 + 0.75 + \cdots + 3 \times (0.5)^9$$

There is a general formula:

> For a geometric sequence with first term a and common ratio r, the sum of the first n terms is:
> $$S_n = \frac{a(1-r^n)}{1-r}$$

So the answer is: $$S_{50} = \frac{3(1-(0.5)^{10})}{1-0.5} = 5.994$$

2.6. To Prove the Formula for a Geometric Series

Write down the series and then multiply it by r:

$$\begin{array}{rcccccccccccl} S_n = & a & + & ar & + & ar^2 & + & ar^3 & + & \ldots & + & ar^{n-1} & \\ rS_n = & & & ar & + & ar^2 & + & ar^3 & + & \ldots & + & ar^{n-1} & + ar^n \end{array}$$

Subtract the second equation from the first:

$$\begin{aligned} S_n - rS_n &= a - ar^n \\ \implies S_n &= \frac{a(1-r^n)}{1-r} \end{aligned}$$

EXERCISES 3.3: **Geometric Sequences and Series**

(1) Find the 8^{th} term and the n^{th} term in the geometric sequence: $5, 10, 20, 40, \ldots$

(2) Find the 15^{th} term and the n^{th} term in the geometric sequence: $-2, 4, -8, 16, \ldots$

(3) In the sequence $1, 3, 9, 27, \ldots$, which is the first term term greater than 1000?

(4) (a) Using the notation above, what are the values of a and r for the sequence: $4, 2, 1, 0.5, 0.25, \ldots$?

(b) Use the formula for a geometric series to calculate: $4 + 2 + 1 + 0.5 + 0.25$.

(5) Find the sum of the first 10 terms of the geometric sequence: $4, 16, 64, \ldots$

(6) Find the sum of the first n terms of the geometric sequence: $20, 4, 0.8, \ldots$ Simplify your answer as much as possible.

(7) Use the formula for a geometric series to show that: $1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} = \frac{16 - x^4}{16 - 8x}$

Further reading and exercises

- For more practice with arithmetic and geometric sequences and series, you could use an A-level pure maths textbook.
- *Jacques* §3.3 Geometric Series.

3. Economic Application: Interest Rates, Savings and Loans

Suppose that you invest £500 at the bank, at a fixed interest rate of 6% (that is, 0.06) per annum, and the interest is paid at the end of each year. At the end of one year you receive an interest payment of $0.06 \times 500 = £30$, which is added to your account, so you have £530. After two years, you receive an interest payment of $0.06 \times 530 = £31.80$, so that you have £561.80 in total, and so on.[1]

More generally, if you invest an amount P (the "principal") and interest is paid annually at interest rate i, then after one year you have a total amount y_1:

$$y_1 = P(1+i)$$

after two years:

$$y_2 = (P(1+i))(1+i) = P(1+i)^2$$

and after t years:

$$y_t = P(1+i)^t$$

This is a geometric sequence with common ratio $(1+i)$.

EXAMPLES 3.1: If you save £500 at a fixed interest rate of 6% paid annually:

(i) How much will you have after 10 years?
Using the formula above, $y_{10} = 500 \times 1.06^{10} = £895.42$.

(ii) How long will you have to wait to double your initial investment?
The initial amount will have doubled when:

$$\begin{aligned} 500 \times (1.06)^t &= 1000 \\ \Longrightarrow (1.06)^t &= 2 \end{aligned}$$

Taking logs of both sides (see chapter 1, section 5):

$$\begin{aligned} t \log_{10} 1.06 &= \log_{10} 2 \\ t &= \frac{\log_{10} 2}{\log_{10} 1.06} = 11.8957 \end{aligned}$$

So you will have to wait 12 years.

3.1. Interval of Compounding

In the previous section we assumed that interest was paid annually. However, in practice, financial institutions often pay interest more frequently, perhaps quarterly or even monthly. We call the time period between interest payments the interval of compounding.

Suppose the bank has a *nominal* (that is, stated) interest rate i, but pays interest m times a year at a rate of $\frac{i}{m}$. After 1 year you would have:

$$P\left(1+\frac{i}{m}\right)^m$$

and after t years:

$$P\left(1+\frac{i}{m}\right)^{mt}$$

[1] If you are not confident with calculations involving percentages, work through *Jacques* Chapter 3.1

EXAMPLES 3.2: You invest £1000 for five years in the bank, which pays interest at a nominal rate of 8%.

(i) How much will you have at the end of one year if the bank pays interest annually? You will have: $1000 \times 1.08 = £1080$.

(ii) How much will you have at the end of one year if the bank pays interest quarterly? Using the formula above with $m = 4$, you will have $1000 \times 1.02^4 = £1082.43$. Note that you are better off (for a given nominal rate) if the interval of compounding is shorter.

(iii) How much will you have at the end of 5 years if the bank pays interest monthly? Using the formula with $m = 12$ and $t = 5$, you will have:

$$1000 \times \left(1 + \frac{0.08}{12}\right)^{5\times12} = £1489.85$$

From this example, you can see that if the bank pays interest quarterly and the nominal rate is 8%, then your investment actually grows by 8.243% in one year. The *effective* annual interest rate is 8.243%. In the UK this rate is known as the *Annual Equivalent Rate (AER)* (or sometimes the *Annual Percentage Rate (APR)*). Banks often describe their savings accounts in terms of the AER, so that customers do not need to do calculations involving the interval of compounding.

If the nominal interest rate is i, and interest is paid m times a year, an investment P grows to $P(1 + i/m)^m$ in one year. So the formula for the Annual Equivalent Rate is:

$$AER = \left(1 + \frac{i}{m}\right)^m - 1$$

EXAMPLES 3.3: *Annual Equivalent Rate*
If the nominal interest rate is 6% and the bank pays interest monthly, what is the AER?

$$AER = \left(1 + \frac{0.06}{12}\right)^{12} - 1 = 0.0617$$

The Annual Equivalent Rate is 6.17%.

3.2. Regular Savings

Suppose that you invest an amount A at the beginning of every year, at a fixed interest rate i (compounded annually). At the end of t years, the amount you invested at the beginning of the first year will be worth $A(1+i)^t$, the amount you invested in the second year will be worth $A(1+i)^{t-1}$, and so on. The total amount that you will have at the end of t years is:

$$\begin{aligned} S_t &= A(1+i)^t + A(1+i)^{t-1} + A(1+i)^{t-2} + \cdots + A(1+i)^2 + A(1+i) \\ &= A(1+i) + A(1+i)^2 + A(1+i)^3 + \cdots + A(1+i)^{t-1} + A(1+i)^t \end{aligned}$$

This is the sum of the first t terms of a geometric sequence with first term $A(1+i)$, and common ratio $(1+i)$. We can use the formula from section 2.5. The sum is:

$$S_t = \frac{A(1+i)\left(1 - (1+i)^t\right)}{1 - (1+i)} = \frac{A(1+i)}{i}\left((1+i)^t - 1\right)$$

So, for example, if you saved £200 at the beginning of each year for 10 years, at 5% interest, then you would accumulate $200\frac{1.05}{0.05}((1.05)^{10} - 1) = £2641.36$.

3.3. Paying Back a Loan

If you borrow an amount L, to be paid back in annual repayments over t years, and the interest rate is i, how much do you need to repay each year?

Let the annual repayment be y. At the end of the first year, interest will have been added to the loan. After repaying y you will owe:

$$X_1 = L(1+i) - y$$

At the end of two years you will owe:

$$\begin{aligned} X_2 &= (L(1+i) - y)(1+i) - y \\ &= L(1+i)^2 - y(1+i) - y \end{aligned}$$

At the end of three years: $X_3 = L(1+i)^3 - y(1+i)^2 - y(1+i) - y$

and at the end of t years: $X_t = L(1+i)^t - y(1+i)^{t-1} - y(1+i)^{t-2} - \ldots - y$

But if you are to pay off the loan in t years, X_t must be zero:

$$\implies L(1+i)^t = y(1+i)^{t-1} + y(1+i)^{t-2} + \ldots + y$$

The right-hand side of this equation is the sum of t terms of a geometric sequence with first term y and common ratio $(1+i)$ (in reverse order). Using the formula from section 2.5:

$$\begin{aligned} L(1+i)^t &= \frac{y\left((1+i)^t - 1\right)}{i} \\ \implies y &= \frac{Li(1+i)^t}{((1+i)^t - 1)} \end{aligned}$$

This is the amount that you need to repay each year.

EXERCISES 3.4: **Interest Rates, Savings, and Loans**
(Assume annual compounding unless otherwise specified.)

(1) Suppose that you save £300 at a fixed interest rate of 4% per annum.
   (a) How much would you have after 4 years if interest were paid annually?
   (b) How much would you have after 10 years if interest were compounded monthly?
(2) If you invest £20 at 15% interest, how long will it be before you have £100?
(3) If a bank pays interest daily and the nominal rate is 5%, what is the AER?
(4) If you save £10 at the beginning of each year for 20 years, at an interest rate of 9%, how much will you have at the end of 20 years?
(5) Suppose you take out a loan of £100000, to be repaid in regular annual repayments, and the annual interest rate is 5%.
   (a) What should the repayments be if the loan is to be repaid in 25 years?
   (b) Find a formula for the repayments if the repayment period is T years.

Further reading and exercises

- *Jacques* §3.1 and 3.2.
- *Anthony & Biggs* Chapter 4

4. Present Value and Investment

Would you prefer to receive (a) a gift of £1000 today, or (b) a gift of £1050 in one year's time?

Your decision (assuming you do not have a desperate need for some immediate cash) will depend on the interest rate. If you accepted the £1000 today, and saved it at interest rate i, you would have £$1000(1+i)$ in a year's time. We could say:

$$\begin{aligned}\text{Future value of (a)} &= 1000(1+i)\\ \text{Future value of (b)} &= 1050\end{aligned}$$

You should accept the gift that has higher future value. For example, if the interest rate is 8%, the future value of (a) is £1080, so you should accept that. But if the interest rate is less than 5%, it would be better to take (b).

Another way of looking at this is to consider what cash sum *now* would be equivalent to a gift of a gift of £1050 in one year's time. An amount P received now would be equivalent to an amount £1050 in one year's time if:

$$\begin{aligned}P(1+i) &= 1050\\ \Longrightarrow P &= \frac{1050}{1+i}\end{aligned}$$

We say that:

The *Present Value* of "£1050 in one year's time" is $\dfrac{1050}{1+i}$

More generally:

> If the annual interest rate is i, the *Present Value* of an amount A to be received in t years' time is:
> $$P = \frac{A}{(1+i)^t}$$

The present value is also known as the *Present Discounted Value*; payments received in the future are worth less – we "discount" them at the interest rate i.

EXAMPLES 4.1: *Present Value and Investment*

(i) The prize in a lottery is £5000, but the prize will be paid in two years' time. A friend of yours has the winning ticket. How much would you be prepared to pay to buy the ticket, if you are able to borrow and save at an interest rate of 5%?

The present value of the ticket is:

$$P = \frac{5000}{(1.05)^2} = 4535.15$$

This is the maximum amount you should pay. If you have £4535.15, you would be indifferent between (a) paying this for the ticket, and (b) saving your money at 5%. Or, if you don't have any money at the moment, you would be indifferent between (a) taking out a loan of £4535.15, buying the ticket, and repaying the loan after 2 years when you receive the prize, and (b) doing nothing. Either way, if your friend will sell the ticket for less than £4535.15, you should buy it.

We can see from this example that *Present Value* is a powerful concept: a single calculation of the PV enables you to answer the question, without thinking about exactly how the money to buy the ticket is to be obtained. This does rely, however, on the assumption that you can borrow and save at the same interest rate.

(ii) An investment opportunity promises you a payment of £1000 at the end of each of the next 10 years, and a capital sum of £5000 at the end of the 11th year, for an initial outlay of £10000. If the interest rate is 4%, should you take it?

We can calculate the present value of the investment opportunity by adding up the present values of all the amounts paid out and received:

$$P = -10000 + \frac{1000}{1.04} + \frac{1000}{1.04^2} + \frac{1000}{1.04^3} + \cdots + \frac{1000}{1.04^{10}} + \frac{5000}{1.04^{11}}$$

In the middle of this expression we have (again) a geometric series. The first term is $\frac{1000}{1.04}$ and the common ratio is $\frac{1}{1.04}$. Using the formula from section 2.5:

$$\begin{aligned} P &= -10000 + \frac{\frac{1000}{1.04}\left(1 - \left(\frac{1}{1.04}\right)^{10}\right)}{1 - \frac{1}{1.04}} + \frac{5000}{1.04^{11}} \\ &= -10000 + \frac{1000\left(1 - \left(\frac{1}{1.04}\right)^{10}\right)}{0.04} + 3247.90 \\ &= -10000 + 25000\left(1 - \left(\frac{1}{1.04}\right)^{10}\right) + 3247.90 \\ &= -10000 + 8110.90 + 3247.90 = -10000 + 11358.80 = £1358.80 \end{aligned}$$

The present value of the opportunity is positive (or equivalently, the present value of the return is greater than the initial outlay): you should take it.

4.1. Annuities

An *annuity* is a financial asset which pays you an amount A each year for N years. Using the formula for a geometric series, we can calculate the present value of an annuity:

$$\begin{aligned} PV &= \frac{A}{1+i} + \frac{A}{(1+i)^2} + \frac{A}{(1+i)^3} + \cdots + \frac{A}{(1+i)^N} \\ &= \frac{\frac{A}{1+i}\left(1 - \left(\frac{1}{1+i}\right)^N\right)}{1 - \left(\frac{1}{1+i}\right)} \\ &= \frac{A\left(1 - \left(\frac{1}{1+i}\right)^N\right)}{i} \end{aligned}$$

The present value tells you the price you would be prepared to pay for the asset.

Exercises 3.5: **Present Value and Investment**

(1) On your 18^{th} birthday, your parents promise you a gift of £500 when you are 21. What is the present value of the gift (a) if the interest rate is 3% (b) if the interest rate is 10%?

(2) (a) How much would you pay for an annuity that pays £20 a year for 10 years, if the interest rate is 5%?

(b) You buy it, then after receiving the third payment, you consider selling the annuity. What price will you be prepared to accept?

(3) The useful life of a bus is five years. Operating the bus brings annual profits of £10000. What is the value of a new bus if the interest rate is 6%?

(4) An investment project requires an initial outlay of £2400, and can generate revenue of £2000 per year. In the first year, operating costs are £600; thereafter operating costs increase by £500 a year.

(a) What is the maximum length of time for which the project should operate?

(b) Should it be undertaken if the interest rate is 5%?

(c) Should it be undertaken if the interest rate is 10%?

Further reading and exercises

- *Jacques* §3.4
- *Anthony & Biggs* Chapter 4
- *Varian* also discusses Present Value and has more economic examples.

5. Limits

5.1. The Limit of a Sequence

If we write down some of the terms of the geometric sequence: $u_n = (\frac{1}{2})^n$:

$$\begin{aligned} u_1 = & \quad (\tfrac{1}{2})^1 & = 0.5 \\ u_{10} = & \quad (\tfrac{1}{2})^{10} & = 0.000977 \\ u_{20} = & \quad (\tfrac{1}{2})^{20} & = 0.000000954 \end{aligned}$$

we can see that as n gets larger, u_n gets closer and closer to zero. We say that "the limit of the sequence as n tends to infinity is zero" or "the sequence converges to zero" or:

$$\lim_{n\to\infty} u_n = 0$$

EXAMPLES 5.1: *Limits of Sequences*

(i) $u_n = 4 - (0.1)^n$
The sequence is:

$$3.9, 3.99, 3.999, 3.9999, \ldots$$

We can see that it converges:

$$\lim_{n\to\infty} u_n = 4$$

(ii) $u_n = (-1)^n$
This sequence is $-1, +1, -1, +1, -1, +1, \ldots$ It has no limit.

(iii) $u_n = \frac{1}{n}$
The terms of this sequence get smaller and smaller:

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \ldots$$

It converges to zero:

$$\lim_{n\to\infty} \tfrac{1}{n} = 0$$

(iv) $2, 4, 8, 16, 32, \ldots$
This is a geometric sequence with common ratio 2. The terms get bigger and bigger. It *diverges*:

$$u_n \to \infty \text{ as } n \to \infty$$

(v) $u_n = \dfrac{2n^3 + n^2}{3n^3}$.

A useful trick is to divide the numerator and the denominator by the highest power of n; that is, by n^3. Then:

$$u_n = \left(\frac{2 + \frac{1}{n}}{3}\right)$$

and we know that $\frac{1}{n} \to 0$, so:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(\frac{2 + \frac{1}{n}}{3}\right) = \frac{2}{3}$$

EXERCISES 3.6: Say whether each if the following sequences converges or diverges as $n \to \infty$. If it converges, find the limit.

(1) $u_n = (\frac{1}{3})^n$

(2) $u_n = -5 + (\frac{1}{4})^n$

(3) $u_n = (-\frac{1}{3})^n$

(4) $u_n = 7 - (\frac{2}{5})^n$

(5) $u_n = \frac{10}{n^3}$

(6) $u_n = (1.2)^n$

(7) $u_n = 25n + \frac{10}{n^3}$

(8) $u_n = \frac{7n^2+5n}{n^2}$

From examples like these we can deduce some general results that are worth remembering:

- $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0$
- Similarly $\displaystyle\lim_{n\to\infty} \frac{1}{n^2} = 0$ and $\displaystyle\lim_{n\to\infty} \frac{1}{n^3} = 0$ etc
- If $|r| < 1$, $\quad \displaystyle\lim_{n\to\infty} r^n = 0$
- If $r > 1$, $\quad r^n \to \infty$

5.2. Infinite Geometric Series

Consider the sequence: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ It is a geometric sequence with first term $a = 1$ and common ratio $r = \frac{1}{2}$. We can find the sum of the first n terms:

$$S_n = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots + \left(\tfrac{1}{2}\right)^{n-1}$$

using the formula from section 2.5:

$$\begin{aligned} S_n &= \frac{a(1-r^n)}{1-r} = \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 2\left(1-\left(\tfrac{1}{2}\right)^n\right) \\ &= 2 - \left(\tfrac{1}{2}\right)^{n-1} \end{aligned}$$

As the number of terms gets larger and larger, their sum gets closer and closer to 2:

$$\lim_{n\to\infty} S_n = 2$$

Equivalently, we can write this as:

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots = 2$$

So we have found the sum of an *infinite* number of terms to be a finite number. Using the sigma notation this can be written:

$$\sum_{i=1}^{\infty} \left(\tfrac{1}{2}\right)^{i-1} = 2$$

or (a little more neatly):

$$\sum_{i=0}^{\infty} \left(\tfrac{1}{2}\right)^i = 2$$

The same procedure works for any geometric series with common ratio r, provided that $|r| < 1$. The sum of the first n terms is:

$$S_n = \frac{a(1-r^n)}{1-r}$$

As $n \to \infty$, $r^n \to 0$ so the series converges:

$$\lim_{n\to\infty} S_n = \frac{a}{1-r}$$

or equivalently:

$$\sum_{i=1}^{\infty} ar^{i-1} = \frac{a}{1-r}$$

But note that if $|r| > 1$ the terms of the series get bigger and bigger, so it *diverges*: the infinite sum does not exist.

Examples 5.2: Find the sum to infinity of the following series:

(i) $2 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} \dots$
This is a geometric series, with $a = 2$ and $r = -\frac{1}{2}$. It converges because $|r| < 1$.
Using the formula above:

$$S_\infty = \frac{a}{1-r} = \frac{2}{\frac{3}{2}} = \frac{4}{3}$$

(ii) $x + 2x^2 + 4x^3 + 8x^4 + \dots$ (assuming $0 < x < 0.5$)
This is a geometric series with $a = x$ and $r = 2x$. We know $0 < r < 1$, so it converges.
The formula for the sum to infinity gives:

$$S_\infty = \frac{a}{1-r} = \frac{x}{1-2x}$$

5.3. Economic Application: Perpetuities

In section 4.1 we calculated the present value of an annuity – an asset that pays you an amount A each year for a fixed number of years. A *perpetuity* is an asset that pays you an amount A each year *forever*.
If the interest rate is i, the present value of a perpetuity is:

$$PV = \frac{A}{1+i} + \frac{A}{(1+i)^2} + \frac{A}{(1+i)^3} + \dots$$

This is an infinite geometric series. The common ratio is $\frac{1}{1+i}$. Using the formula for the sum of an infinite series:

$$\begin{aligned} PV &= \frac{\frac{A}{1+i}}{1 - \left(\frac{1}{1+i}\right)} \\ &= \frac{A}{i} \end{aligned}$$

Again, the present value tells you the price you would be prepared to pay for the asset. Even an asset that pays out forever has a finite price. (Another way to get this result is to let

$N \to \infty$ in the formula for the present value of an annuity that we obtained earlier.)

Exercises 3.7: **Infinite Series, and Perpetuities**

(1) Evaluate the following infinite sums:

(a) $\frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4 + \dots$

(b) $1 + 0.2 + 0.04 + 0.008 + 0.0016 + \dots$

(c) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \dots$

(d) $\frac{2}{3} + \left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^7 + \dots$

(e) $\sum_{r=3}^{\infty} \left(\frac{1}{2}\right)^r$

(f) $\sum_{r=0}^{\infty} x^r$ assuming $|x| < 1$. (Why is this assumption necessary?)

(g) $\frac{2y^2}{x} + \frac{4y^3}{x^2} + \frac{8y^4}{x^3} + \dots$ What assumption is needed here?

(2) If the interest rate is 4%, what is the present value of:

(a) an annuity that pays £100 each year for 20 years?

(b) a perpetuity that pays £100 each year forever?

How will the value of each asset have changed after 10 years?

(3) A firm's profits are expected to be £1000 this year, and then to rise by 2% each year after that (forever). If the interest rate is 5%, what is the present value of the firm?

Further reading and exercises

- *Anthony & Biggs*: §3.3 discusses limits briefly.
- *Varian* has more on financial assets including perpetuities, and works out the present value of a perpetuity in a different way.
- For more on limits of sequences, and infinite sums, refer to an A-level pure maths textbook.

6. The Number e

If we evaluate the numbers in the sequence:

$$u_n = \left(1 + \frac{1}{n}\right)^n$$

we get: $u_1 = 2$, $u_2 = (1 + \frac{1}{2})^2 = 2.25$, $u_3 = (1 + \frac{1}{3})^3 = 2.370, \ldots$
For some higher values of n we have, for example:

$$\begin{aligned} u_{10} &= & (1.1)^{10} & = 2.594 \\ u_{100} &= & (1.01)^{100} & = 2.705 \\ u_{1000} &= & (1.001)^{1000} & = 2.717 \\ u_{10000} &= & (1.0001)^{10000} & = 2.71814 \\ u_{100000} &= & (1.00001)^{100000} & = 2.71826 \quad \ldots \end{aligned}$$

As $n \to \infty$, u_n gets closer and closer to a limit of $2.718281828459\ldots$ This is an irrational number (see Chapter 1) known simply as e. So:

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \quad (\approx 2.71828)$$

e is important in calculus (as we will see later) and arises in many economic applications.
We can generalise this result to:

> For any value of r, $\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n = e^r$

Exercises 3.8: Verify (approximately, using a calculator) that $\lim_{n\to\infty} \left(1 + \frac{2}{n}\right)^n = e^2$.
Hint: Most calculators have a button that evaluates e^x for any number x.

6.1. Economic Application: Continuous Compounding

Remember, from section 3.1, that if interest is paid m times a year and the nominal rate is i, then the return after t years from investing an inital amount P is:

$$P\left(1 + \frac{i}{m}\right)^{mt}$$

Interest might be paid quarterly ($m = 4$), monthly ($m = 12$), weekly ($m = 52$), or daily ($m = 365$). Or it could be paid even more frequently – every hour, every second ... As the interval of compounding get shorter, interest is compounded almost continuously.
As $m \to \infty$, we can apply our result above to say that:

$$\lim_{m\to\infty} \left(1 + \frac{i}{m}\right)^m = e^i$$

and so:

> If interest is compounded continuously at rate i, the return after t years on an initial amount P is:
> Pe^{it}

EXAMPLES 6.1: If interest is compounded continuously, what is the AER if:

(i) the interest rate is 5%?
Applying the formula, an amount P invested for one year yields:

$$Pe^{0.05} = 1.05127P = (1 + 0.05127)P$$

So the AER is 5.127%.

(ii) the interest rate is 8%?
Similarly, $e^{0.08} = 1.08329$, so the AER is 8.329%.

We can see from these examples that with continuous compounding the AER is little different from the interest rate. So, when solving economic problems we often simplify by assuming continuous compounding, because it avoids the messy calculations for the interval of compounding.

6.2. Present Value with Continuous Compounding

In section 4, when we showed that the present value of an amount A received in t years time is $\frac{A}{(1+i)^t}$, we were assuming annual compounding of interest.

> With continuous compounding, if the interest rate is i, the present value of an amount A received in t years is:
> $P = Ae^{-it}$

Continuous compounding is particularly useful because it allows us to calculate the present value when t is not a whole number of years.

To see where the formula comes from, note that if you have an amount Ae^{-it} now, and you save it for t years with continuous compounding, you will then have $Ae^{-it}e^{it} = A$. So "Ae^{-it} now" and "A after t years", are worth the same.

EXERCISES 3.9: e

(1) Express the following in terms of e:

(a) $\lim_{n\to\infty}(1 + \frac{1}{n})^n$ (b) $\lim_{n\to\infty}(1 + \frac{5}{n})^n$ (c) $\lim_{n\to\infty}(1 + \frac{1}{2n})^n$

(2) If you invest £100, the interest rate is 5%, and interest is compounded countinuously:

(a) How much will you have after 1 year?

(b) How much will you have after 5 years?

(c) What is the AER?

(3) You expect to receive a gift of £100 on your next birthday. If the interest rate is 5%, what is the present value of the gift (a) six months before your birthday (b) 2 days before your birthday?

Further reading and exercises

- *Anthony & Biggs*: §7.2 and §7.3.
- *Jacques* §2.4.

Solutions to Exercises in Chapter 3

EXERCISES 3.1:

(1) (a) 14
 (b) 2.75
 (c) -13
 (d) 2.25
 (e) -6

(2) $-6, 6, 26$

(3) $4, 4, 4, 4, 4$

(4) $u_4 = -2$

(5) 30

(6) (a) $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$
 (b) $1 + 2 + 4 + 8$
 (c) $1+3+5+\ldots+(2n+1)$

(7) (a) n
 (b) $4n - 3$

(8) There are several possibilities. e.g.
 (a) $\sum_{r=1}^{25} r^2$
 (b) $\sum_{r=2}^{7} 3r$
 (c) $\sum_{r=4}^{n} r^2$

EXERCISES 3.2:

(1) (a) $a = 4$ $d = 3$
 (b) $a = -1$ $d = 3$
 (c) $a = -7$ $d = -1.5$

(2) (a) $u_n = 44 + 2(n-1)$ $= 42 + 2n$
 (b) $u_n = -3 - 4(n-1)$ $= -4n + 1$

(3) $u_1 = 24$

(4) 288

(5) (a) $a = 21$ $d = -2$
 (b) $S_5 = 85$
 (c) $S_{11} = 121$

(6) Yes, $u_n = -2 - 4(n-1)$, so $d = -4$

EXERCISES 3.3:

(1) 640, $5 \times 2^{n-1}$

(2) -32768, $(-2)^n$

(3) 8^{th}

(4) (a) $a = 4$, $r = \frac{1}{2}$
 (b) $S_5 = 7.75$

(5) $S_{10} = 1398100$

(6) $S_n = 25(1 - \frac{1}{5}^n)$ $= 25 - \left(\frac{1}{5}\right)^{n-2}$

(7) $a = 1; r = \left(\frac{x}{2}\right); n = 4$ $\Rightarrow S_4 = \frac{16-x^4}{16-8x}$

EXERCISES 3.4:

(1) (a) 350.96
 (b) 447.25

(2) $t = 12$

(3) 5.13%

(4) $S_{20} = £557.65$

(5) (a) £7095.25
 (b) $y = \frac{5000(1.05)^T}{1.05^T - 1}$

EXERCISES 3.5:

(1) (a) £457.57
 (b) £375.66

(2) (a) £154.43
 (b) £115.73

(3) £42123

(4) (a) 3 years.
 (b) Yes PV=£95.19
 (c) No PV=−£82.95

EXERCISES 3.6:

(1) $u_n \to 0$ as $n \to \infty$

(2) $u_n \to -5$ as $n \to \infty$

(3) $u_n \to 0$ as $n \to \infty$

(4) $u_n \to 7$ as $n \to \infty$

(5) $u_n \to 0$ as $n \to \infty$

(6) $u_n \to \infty$ as $n \to \infty$

(7) $u_n \to \infty$ as $n \to \infty$

(8) $u_n \to 7$ as $n \to \infty$

EXERCISES 3.7:

(1) (a) $\frac{1}{2}$
 (b) $\frac{5}{4}$
 (c) $\frac{2}{3}$
 (d) $\frac{18}{19}$
 (e) $\frac{1}{4}$
 (f) $\frac{1}{1-x}$ If $|x| \geq 1$ sequence diverges
 (g) $\frac{2y^2}{x-2y}$ assuming $|2y| < |x|$

(2) (a) £1359.03. Is only worth £811.09 after ten years.
 (b) £2500. Same value in ten years.

(3) £33,333.33

EXERCISES 3.8:

(1) ($e^2 = 7.389$ to 3 decimal places)

EXERCISES 3.9:

(1) (a) e
 (b) e^5
 (c) $e^{\frac{1}{2}}$

(2) (a) £105.13
 (b) £128.40
 (c) 5.13%

(3) (a) $P = 100e^{-\frac{i}{2}}$ $= £97.53$
 (b) $P = 100e^{-\frac{2i}{365}}$ $= £99.97$

[2]This Version of Workbook Chapter 3: September 25, 2014

Worksheet 3: Sequences, Series, and Limits; the Economics of Finance

Quick Questions

(1) What is the n^{th} term of each of the following sequences:
(a) $20, 15, 10, 5, \ldots$ (b) $1, 8, 27, 64, \ldots$ (c) $0.2, 0.8, 3.2, 12.8, \ldots$

(2) Write out the series: $\sum_{r=0}^{n-1}(2r-1)^2$ without using sigma notation, showing the first four terms and the last two terms.

(3) For each of the following series, work out how many terms there are and hence find the sum:
(a) $3+4+5+\cdots+20$ (b) $1+0.5+0.25+\cdots+(0.5)^{n-1}$ (c) $5+10+20+\cdots+5\times 2^n$

(4) Express the series $3+7+11+\cdots+(4n-1)+(4n+3)$ using sigma notation.

(5) If you invest £500 at a fixed interest rate of 3% per annum, how much will you have after 4 years:
(a) if interest is paid annually?
(b) if interest is paid monthly? What is the AER in this case?
(c) if interest is compounded continuously?
If interest is paid annually, when will your savings exceed £600?

(6) If the interest rate is 5% per annum, what is the present value of:
(a) An annuity that pays £100 a year for 20 years?
(b) A perpetuity that pays £50 a year?

(7) Find the limit, as $n \to \infty$, of:
(a) $3(1+(0.2)^n)$ (b) $\dfrac{5n^2+4n+3}{2n^2+1}$ (c) $0.75+0.5625+\cdots+(0.75)^n$

Longer Questions

(1) Carol (an economics student) is considering two possible careers. As an acrobat, she will earn £30000 in the first year, and can expect her earnings to increase at 1% per annum thereafter. As a beekeeper, she will earn only £20000 in the first year, but the subsequent increase will be 5% per annum. She plans to work for 40 years.
(a) If she decides to be an acrobat:
(i) How much will she earn in the 3^{rd} year of her career?
(ii) How much will she earn in the n^{th} year?
(iii) What will be her total career earnings?
(b) If she decides to be a beekeeper:
(i) What will be her total career earnings?
(ii) In which year will her annual earnings first exceed what she would have earned as an acrobat?
(c) She knows that what matters for her choice of career is the present value of her earnings. The rate of interest is i. (Assume that earnings are received at the end of each year, and that her choice is made on graduation day.) If she decides to be an acrobat:

(i) What is the present value of her first year's earnings?
(ii) What is the present value of the n^{th} year's earnings?
(iii) What is the total present value of her career earnings?

(d) Which career should she choose if the interest rate is 3%?
(e) Which career should she choose if the interest rate is 15%?
(f) Explain these results.

(2) Bill is due to start a four year degree course financed by his employer and he feels the need to have his own computer. The computer he wants cost £1000. The insurance premium on the computer will start at £40 for the first year, and decline by £5 per year throughout the life of the computer, while repair bills start at £50 in the computer's first year, and increase by 50% per annum thereafter. (The grant and the insurance premium are paid at the beginning of each year, and repair bills at the end.)
The resale value for computers is given by the following table:

	Resale value at end of year
Year 1	75% of initial cost
Year 2	60% of initial cost
Year 3	20% of initial cost
Year 4 and onwards	£100

The interest rate is 10% per annum.

(a) Bill offers to forgo £360 per annum from his grant if his employer purchases a computer for him and meets all insurance and repair bills. The computer will be sold at the end of the degree course and the proceeds paid to the employer. Should the employer agree to the scheme? If not, what value of computer would the employer agree to purchase for Bill (assuming the insurance, repair and resale schedules remain unchanged)?

(b) Bill has another idea. He still wants the £1000 computer, but suggests that it be replaced after two years with a new one. Again, the employer would meet all bills and receive the proceeds from the sale of both computers, and Bill would forgo £360 per annum from his grant. How should the employer respond in this case?

CHAPTER 4

Functions

> Functions, and the language of functions, are widely used in economics. Linear and quadratic functions were discussed in Chapter 2. Now we introduce other useful functions (including **exponential** and **logarithmic** functions) and concepts (such as **inverse functions** and **functions of several variables**). Economic applications are **supply and demand functions**, **utility functions** and **production functions**.

1. Function Notation, and Some Common Functions

We have already encountered some functions in Chapter 2. For example:

$$y = 5x - 8$$

Here y is a function of x (or in other words, y depends on x).

$$C = 3 + 2q^2$$

Here, a firm's total cost, C, of producing output is a function of the quantity of output, q, that it produces.

To emphasize that y *is a function of* x, and that C *is a function of* q, we often write:

$$y(x) = 5x - 8 \quad \text{and} \quad C(q) = 3 + 2q^2$$

Also, for general functions it is common to use the letter f rather than y:

$$f(x) = 5x - 8$$

Here, x is called the *argument* of the function.

In general, we can think of a function $f(x)$ as a "black box" which takes x as an input, and produces an output $f(x)$:

$$x \longrightarrow \boxed{\quad f \quad} \longrightarrow f(x)$$

For each input value, there is a unique output value.

EXAMPLES 1.1: For the function $f(x) = 5x - 8$

(i) Evaluate $f(3)$

$$f(3) = 15 - 8 = 7$$

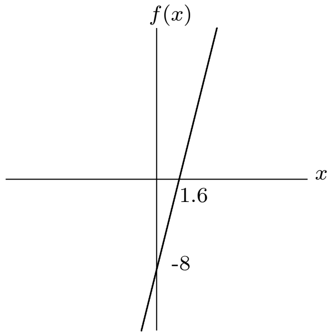
(ii) Evaluate $f(0)$

$$f(0) = 0 - 8 = -8$$

(iii) Solve the equation $f(x) = 0$

$$\begin{aligned} f(x) &= 0 \\ \implies 5x - 8 &= 0 \\ \implies x &= 1.6 \end{aligned}$$

(iv) Hence sketch the graph.



> EXERCISES 4.1: **Using Function Notation**
>
> (1) For the function $f(x) = 9 - 2x$, evaluate $f(2)$ and $f(-4)$.
>
> (2) Solve the equation $g(x) = 0$, where $g(x) = 5 - \frac{10}{x+1}$.
>
> (3) If a firm has cost function $C(q) = q^3 - 5q$, what are its costs of producing 4 units of output?

1.1. Polynomials

Polynomials were introduced in Chapter 1. They are functions of the form:

$$f(x) = a_n x^n + a_{n-1} x^{n+1} + ... + a_0$$

EXAMPLES 1.2:

(i) $g(x) = 5x^3 - 2x + 1$ is a polynomial of degree 3

(ii) $h(x) = 5 + x^4 - x^9 + x^2$ is a polynomial of degree 9

A linear function is a polynomial of degree 1, and a quadratic function is a polynomial of degree 2. We already know what their graphs look like (see Chapter 2); a linear function crosses the x-axis once (or not at all); a quadratic function crosses the x-axis twice, or touches it once, or not at all.

> The graph of a polynomial of degree n crosses the x-axis up to n times.

EXERCISES 4.2: **Polynomials**

(1) For the polynomial function $f(x) = x^3 - 3x^2 + 2x$:

(a) Factorise the function and hence show that it crosses the x-axis at $x = 0$, $x = 1$ and $x = 2$.

(b) Check whether the function is positive or negative when $x < 0$, when $0 < x < 1$, when $1 < x < 2$, and when $x > 2$ and hence sketch a graph of the function.

(2) Consider the polynomial function $g(x) = 5x^2 - x^4 - 4$.

(a) What is the degree of the polynomial?

(b) Factorise the function. (*Hint*: it is a quadratic in x^2: $-(x^2)^2 + 5x^2 - 4$)

(c) Hence sketch the graph, using the same method as in the previous example.

1.2. The Function $f(x) = x^n$

When n is a positive integer, $f(x) = x^n$ is just a simple polynomial. But n could be negative:

$$f(x) = \frac{1}{x^2} \quad (n = -2)$$

and in economic applications, we often use fractional values of n, which we can do if x represents a positive variable (such as "output", or "employment"):

$$\begin{aligned} f(x) &= x^{0.3} \\ f(x) &= x^{\frac{3}{2}} \\ f(x) &= \sqrt{x} \quad (n = 0.5) \end{aligned}$$

EXERCISES 4.3: **The Function x^n**
On a single diagram, with x-axis from 0 to 4, and vertical axis from 0 to 6.5, plot carefully the graphs of $f(x) = x^n$, for n=1.3, n=1, n=0.7 and $n = -0.3$.
Hint: You cannot evaluate $x^{-0.3}$ when $x = 0$, but try values of x close to zero.

You should find that your graphs have the standard shapes below.

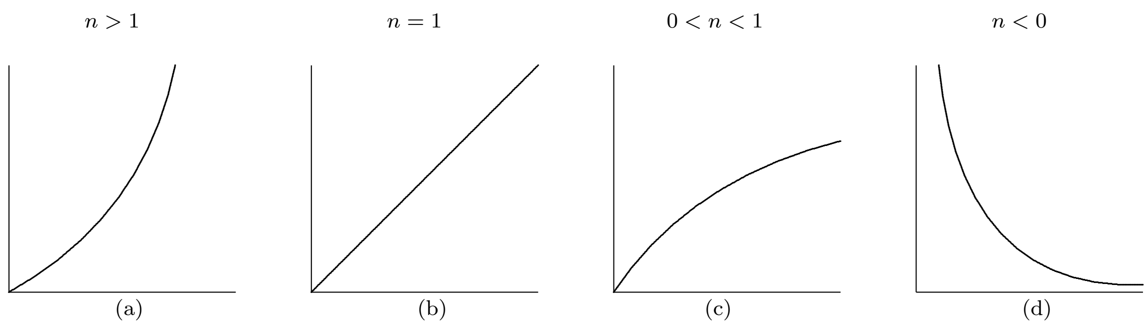


FIGURE 1. The function $f(x) = x^n$

1.3. Increasing and Decreasing Functions

If $f(x)$ increases whenever x increases:

- the graph is upward-sloping;
- we say that f is an *increasing function* of x
  (or sometimes that it is a *monotonic increasing function*).

Similarly, a function that decreases whenever x increases has a downward-sloping graph and is known as a (monotonic) *decreasing function*. *Monotonic* means simply that f moves in one direction (either up or down) as x increases.

EXAMPLES 1.3: $f(x) = x^n \quad (x \geq 0)$
We can see from Figure 1 that this function is monotonic, whatever the value of n.

(i) When $n > 0$ f is an increasing function of x (graphs (a) to (c))

(ii) When $n < 0$ f is a decreasing function of x (graph (d))

1.4. Limits of Functions

In Chapter 3, we looked at limits of sequences. We also use this idea for functions.

EXAMPLES 1.4: $f(x) = x^n$
Looking at Figure 1 again:

(i) If $n > 0$, $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to 0} x^n = 0$

(ii) If $n < 0$, $\lim_{x\to\infty} x^n = 0$ and $\lim_{x\to 0} x^n = \infty$

EXERCISES 4.4: **Increasing and decreasing functions, and limits**

(1) If $f(x) = 2 - x^2$ find:

  (a) $\lim_{x\to\infty} f(x)$

  (b) $\lim_{x\to-\infty} f(x)$

  (c) $\lim_{x\to 0} f(x)$

(2) If $y = \frac{2}{x} + 3$ for values of $x \geq 0$, is y an increasing or a decreasing function of x? What is the limit of y as x tends to infinity?

(3) If $g(x) = 1 - \frac{5}{x^2}$ for $x \geq 0$:

  (a) Is g an increasing or a decreasing function?

  (b) What is $\lim_{x\to\infty} g(x)$?

  (c) What is $\lim_{x\to 0} g(x)$?

Further reading and exercises

- *Jacques* §1.3.
- *Anthony & Biggs* §2.2.
- Both of the above are brief. For more examples, use an A-level pure maths textbook.

2. Composite Functions

If we have two functions, $f(x)$ and $g(x)$, we can take the output of f and input it to g:

$$x \longrightarrow \boxed{\quad f \quad} \longrightarrow f(x) \longrightarrow \boxed{\quad g \quad} \longrightarrow g(f(x))$$

We can think of the final output as the output of a new function, $g(f(x))$, called a *composite function* or a "function of a function."

Examples 2.1: If $f(x) = 2x + 3$ and $g(x) = x^2$:

(i) $g(f(2)) = g(7) = 49$

(ii) $f(g(2)) = f(4) = 11$

(iii) $f(g(-4)) = f(16) = 35$

(iv) $g(f(-4)) = g(-5) = 25$
Note that $f(g(x))$ is not the same as $g(f(x))$!

Examples 2.2: If $f(x) = 2x + 3$ and $g(x) = x^2$, what are the functions:

(i) $f(g(x))$?

$$f(g(x)) = 2g(x) + 3 = 2x^2 + 3$$

(ii) $g(f(x))$?

$$g(f(x)) = (f(x))^2 = (2x + 3)^2 = 4x^2 + 12x + 9$$

Note that we can check these answers using the previous ones. For example:

$$f(g(x)) = 2x^2 + 3 \implies f(g(2)) = 11$$

Exercises 4.5: **Composite Functions**

(1) If $f(x) = 3 - 2x$ and $g(x) = 8x - 1$ evaluate $f(g(2))$ and $g(f(2))$.

(2) If $g(x) = \frac{4}{x+1}$ and $h(x) = x^2 + 1$, find $g(h(1))$ and $h(g(1))$.

(3) If $k(x) = \sqrt[3]{x}$ and $m(x) = x^3$, evaluate $k(m(3))$ and $m(k(3))$. Why are the answers the same in this case?

(4) If $f(x) = x + 1$, and $g(x) = 2x^2$, what are the functions $g(f(x))$, and $f(g(x))$?

(5) If $h(x) = \frac{5}{x+2}$ and $k(x) = \frac{1}{x}$ find the functions $h(k(x))$, and $k(h(x))$.

Further reading and exercises

- *Jacques* §1.3.
- *Anthony & Biggs* §2.3.
- Both of the above are brief. For more examples, use an A-level maths pure textbook.

3. Inverse Functions

Suppose we have a function $f(x)$, and we call the output y. If we can find another function that takes y as an input and produces the original value x as output, it is called the inverse function $f^{-1}(y)$.

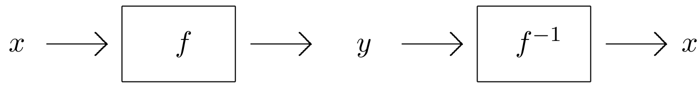


FIGURE 2. f^{-1} is the inverse of f: $f^{-1}(f(x)) = x$

If we can find such a function, and we take its output and input it to the original function, we find also that f is the inverse of f^{-1}:

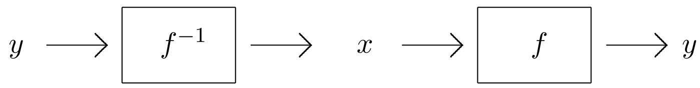


FIGURE 3. f is the inverse of f^{-1}: $f(f^{-1}(y)) = y$

EXAMPLES 3.1: What is the inverse of the function

(i) $f(x) = 3x + 1$?

Call the output of the function y:

$$y = 3x + 1$$

This equation tells you how to find y if you know x (that is, it gives y in terms of x). Now rearrange it, to find x in terms of y (that is, make x the subject):

$$\begin{aligned} 3x &= y - 1 \\ x &= \frac{y-1}{3} \end{aligned}$$

So the inverse function is:

$$f^{-1}(y) = \frac{y-1}{3}$$

(ii) $f(x) = \frac{2}{x-1}$?

$$\begin{aligned} y &= \frac{2}{x-1} \\ y(x-1) &= 2 \\ x &= 1 + \frac{2}{y} \end{aligned}$$

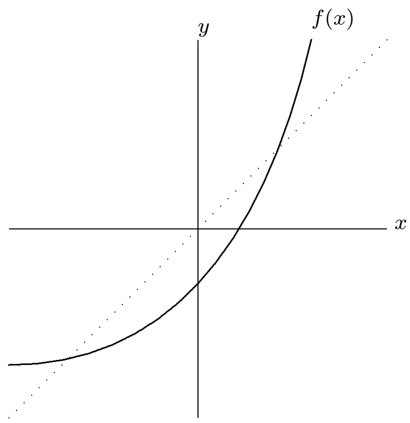
So the inverse function is:

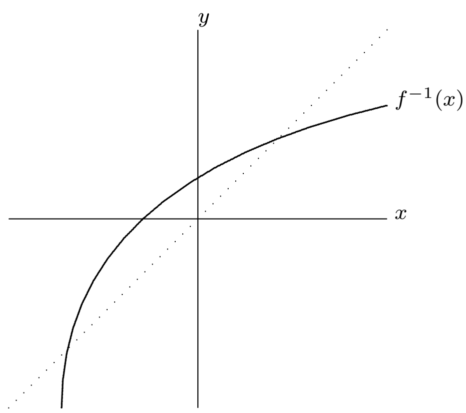
$$f^{-1}(y) = 1 + \frac{2}{y}$$

- It doesn't matter what letter we use for the argument of a function. The function above would still be the same function if we wrote it as:
$$f^{-1}(z) = 1 + \frac{2}{z}$$
- We could say, for example, that
$$f^{-1}(x) = 1 + \frac{2}{x} \text{ is the inverse of } f(x) = \frac{2}{x-1}$$
using the same argument for both (although this can be be confusing).

- *Warning:* It is usually only possible to find the inverse if the function is monotonic (see section 1.3). Otherwise, if you know the output, you can't be sure what the input was. For example, think about the function $y = x^2$. If we know that the output is 9, say, we can't tell whether the input was 3 or -3. In such cases, we say that the inverse function "doesn't exist."

- There is an easy way to work out what the graph of the inverse function looks like: just reflect it in the line $y = x$.





Exercises 4.6: Find the inverse of each of the following functions:

(1) $f(x) = 8x + 7$

(2) $g(x) = 3 - 0.5x$

(3) $h(x) = \frac{1}{x+4}$

(4) $k(x) = x^3$

Further reading and exercises

- *Jacques* §1.3.
- *Anthony & Biggs* §2.2.
- A-level pure maths textbooks.

4. Economic Application: Supply and Demand Functions

4.1. Demand

The market demand function for a good tells us how the quantity that consumers want to buy depends on the price. Suppose the demand function in a market is:

$$Q^d(P) = 90 - 5P$$

This is a linear function of P. You can see that it is downward-sloping (it is a decreasing function). If the price increases, consumers will buy less.

The inverse demand function tells us how much consumers will pay if the quantity available is Q. To find the inverse demand function:

$$\begin{aligned} Q &= 90 - 5P \\ 5P &= 90 - Q \\ P &= \frac{90 - Q}{5} \end{aligned}$$

So the inverse demand function is:

$$P^d(Q) = \frac{90 - Q}{5}$$

4.2. Supply

Suppose the supply function (the quantity that firms are willing to supply if the market price is P) is:

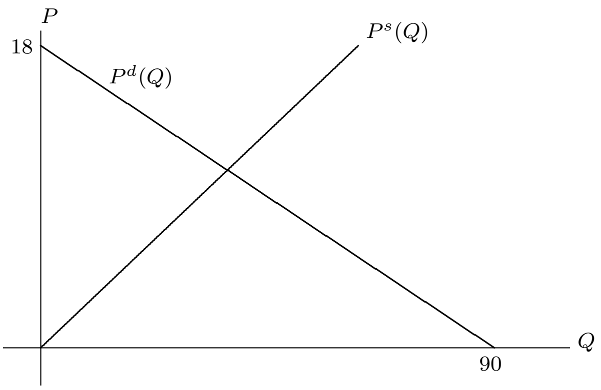
$$Q^s(P) = 4P$$

This is an increasing function (upward-sloping). Firms will supply more if the price is higher. The inverse supply function is:

$$P^s(Q) = \frac{Q}{4}$$

4.3. Market Equilibrium

We can draw the supply and demand functions to show the equilibrium in the market. It is conventional to draw the graph of P against Q – that is, to put P on the vertical axis, and draw the *inverse* supply and demand functions).



The equilibrium in the market is where the supply price equals the demand price:

$$\begin{aligned} P^s(Q) &= P^d(Q) \\ \Longrightarrow \frac{Q}{4} &= \frac{90 - Q}{5} \\ 5Q &= 360 - 4Q \\ Q &= 40 \end{aligned}$$

and so:

$$P = 10$$

EXERCISES 4.7: Suppose that the supply and demand functions in a market are:

$$Q^s(P) = 6P - 10$$
$$Q^d(P) = \frac{100}{P}$$

(1) Find the inverse supply and demand functions, and sketch them.

(2) Find the equilibrium price and quantity in the market.

4.4. Using Parameters to Specify Functions

In economic applications, we often want to specify the general shape of a function, without giving its exact formula. To do this, we can include *parameters* in the function.

For example, in the previous section, we used a particular demand function:

$$P^d(Q) = \frac{90 - Q}{5}$$

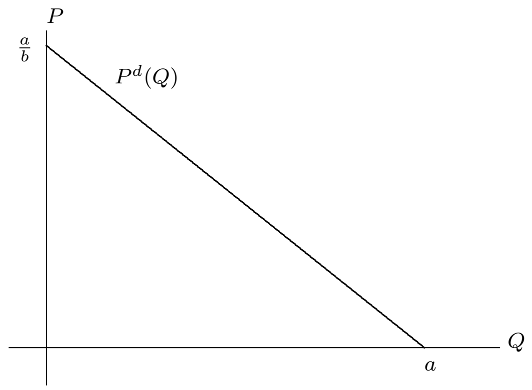
A more general specification would be to write it using two parameters a and b, instead of the numbers:

$$P^d(Q) = \frac{a - Q}{b} \text{ where } a > 0 \text{ and } b > 0$$

This gives us enough information to sketch the general shape of the function:

- it is a downward-sloping straight line (if we write it as
$$P = -\frac{1}{b}Q + \frac{a}{b}$$
we can see that the gradient is negative);

- and we can find the points where it crosses the axes.



EXERCISES 4.8: Suppose that the inverse supply and demand functions in a market are:

$$P^d(Q) = a - Q$$
$$P^s(Q) = cQ + d \quad \text{where } a, c, d > 0$$

(1) Sketch the functions.

(2) Find the equilibrium quantity in terms of the parameters a, c and d.

(3) What happens if $d > a$?

Further reading and exercises

- *Jacques* §1.3.
- *Anthony & Biggs* §1.

5. Exponential and Logarithmic Functions

5.1. Exponential Functions

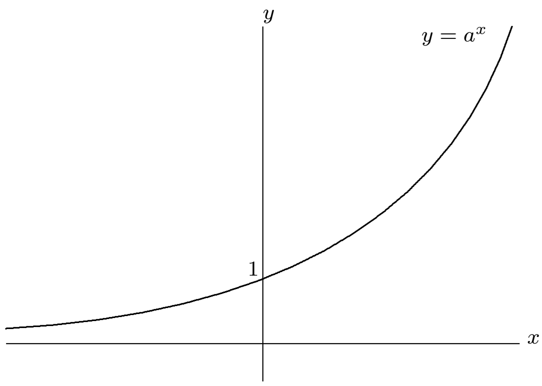
$$f(x) = a^x$$

where a is any positive number, is called an *exponential function.* For example, 3^x and 8.31^x are both exponential functions.

Exponential functions have the same general shape, whatever the value of a. If $a > 1$:

- the y-intercept is 1, because $a^0 = 1$;
- y is positive, and increasing, for all values of x;
- as x gets bigger, y increases very fast (exponentially);
- as x gets more negative, y gets closer to zero (but never actually gets there):

$$\lim_{x \to -\infty} a^x = 0$$



(If $a < 1$, a^x is a decreasing function.)

5.2. Logarithmic Functions

$$g(x) = \log_a x$$

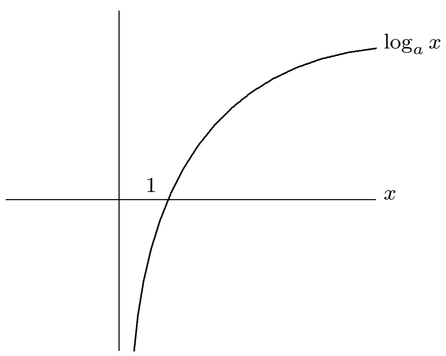
is a logarithmic function, with base a. Remember, from the definition of a logarithm in Chapter 1, that:

$$z = \log_a x \quad \text{is equivalent to} \quad x = a^z$$

In other words, a logarithmic function is the inverse of an exponential function.

Since a logarithmic function is the inverse of an exponential function, we can find the shape of the graph by reflecting the graph above in the line $y = x$. We can see that if $a > 1$:

- Since a^z is always positive, the logarithmic function is only defined for positive values of x.
- It is an increasing function
- Since $a^0 = 1$, $\log_a(1) = 0$
- $\log_a x < 0$ when $0 < x < 1$
- $\log_a x > 0$ when $x > 1$
- $\lim_{x \to 0} \log_a x = -\infty$



EXERCISES 4.9: **Exponential and Logarithmic Functions**

(1) Plot the graph of the exponential function $f(x) = 2^x$, for $-3 \leq x \leq 3$.

(2) What is (a) $\log_2 2$ (b) $\log_a a$?

(3) Plot the graph of $y = \log_{10} x$ for values of x between 0 and 10.
Hint: The "log" button on most calculators gives you $\log_{10}$.

5.3. *The* Exponential Function

In Chapter 3 we came across the number e: $e \approx 2.71828$. The function

$$e^x$$

is known as *the* exponential function.

5.4. Natural Logarithms

The inverse of the exponential function is the *natural logarithm function*. We could write it as:

$$\log_e x$$

but it is often written instead as:

$$\ln x$$

Note that since e^x and ln are inverse functions:

$$\ln(e^x) = x \text{ and } e^{\ln x} = x$$

5.5. Where the Exponential Function Comes From

Remember (or look again at Chapter 3) where e comes from: it is the limit of a sequence of numbers. For any number r:

$$e^r = \lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n$$

So, we can think of the exponential function as the limit of a sequence of functions:

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

EXERCISES 4.10: **Base** e

(1) Plot the graphs of $1 + x$, $\left(1 + \frac{x}{2}\right)^2$, $\left(1 + \frac{x}{3}\right)^3$, and e^x, for $0 \leq x \leq 1$.

(2) What is (i) $\ln 1$ (ii) $\ln e$ (iii) $\ln\left(e^{5x}\right)$ (iv) $e^{\ln x^2}$ (v) $e^{\ln 3 + \ln x}$?

(3) Plot the graph of $y = \ln x$ for values of x between 0 and 3.

(4) Sketch the graph of $y = e^x$. From this sketch, work out how to sketch the graphs of $y = e^{-x}$ and $y = e^{3x+1}$.
Hint: There is an *ln* button, and an e^x button, on most calculators.

Further reading and exercises

- *Jacques* §2.4.
- *Anthony & Biggs* §7.1 to §7.4.

6. Economic Examples using Exponential and Logarithmic Functions

From Chapter 3 we know that an initial amount A_0 invested at interest rate i with continuous compounding of interest would grow to be worth

$$A(t) = A_0 e^{it}$$

after t years. We can think of this as an exponential function of time. Exponential functions are used to model the growth of other economic variables over time:

EXAMPLES 6.1: A company selling cars expects to increase its sales over future years. The number of cars sold per day after t years is expected to be:

$$S(t) = 5e^{0.08t}$$

(i) How many cars are sold per day now?
When $t = 0$, $S = 5$.

(ii) What is the expected sales rate after (a) 1 year; (b) 5 years (c) 10 years?
$S(1) = 5e^{0.08} = 5.4$.
Similarly, (b) $S(5) = 7.5$ and (c) $S(10) = 11.1$.

(iii) Sketch the graph of the sales rate against time.

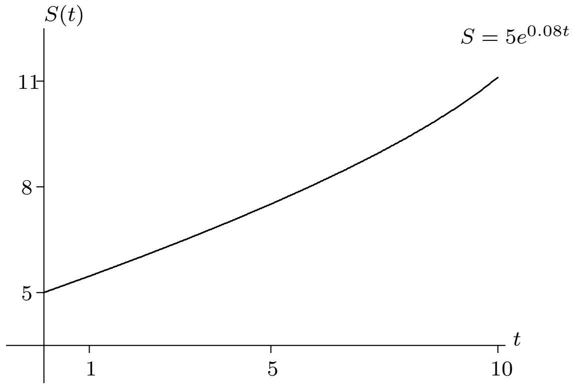
(iv) When will daily sales first exceed 14?

To find the time when $S = 14$, we must solve the equation:

$$14 = 5e^{0.08t}$$

Take (natural) logs of both sides:

$$\begin{aligned} \ln 14 &= \ln 5 + \ln(e^{0.08t}) \\ \ln 14 - \ln 5 &= 0.08t \\ 1.03 &= 0.08t \\ t &= 12.9 \end{aligned}$$



EXAMPLES 6.2: Y is the GDP of a country. GDP in year t satisfies (approximately) the equation:

$$Y = ae^{0.03t}$$

(i) What is GDP in year 0?
Putting $t = 0$ in the equation, we obtain: $Y(0) = a$. So the parameter a represents the initial value of GDP.

(ii) What is the percentage change in GDP between year 5 and year 6?
$Y(5) = ae^{0.15} = 1.1618a$ and $Y(6) = ae^{0.18} = 1.1972a$

So the percentage increase in GDP is:

$$100 \times \frac{Y(6) - Y(5)}{Y(5)} = 100\frac{.0354a}{1.1618a} = 3.05$$

The increase is approximately 3 percent.

(iii) What is the percentage change in GDP between year t and year $t + 1$?

$$\begin{aligned} 100 \times \frac{Y(t+1) - Y(t)}{Y(t)} &= 100\frac{ae^{0.03(t+1)} - ae^{0.03t}}{ae^{0.03t}} \\ &= 100\frac{ae^{0.03t}(e^{0.03} - 1)}{ae^{0.03t}} \\ &= 100(e^{0.03} - 1) \\ &= 3.05 \end{aligned}$$

So the percentage increase is approximately 3 percent in *every* year.

(iv) Show that the graph of $\ln Y$ against time is a straight line.

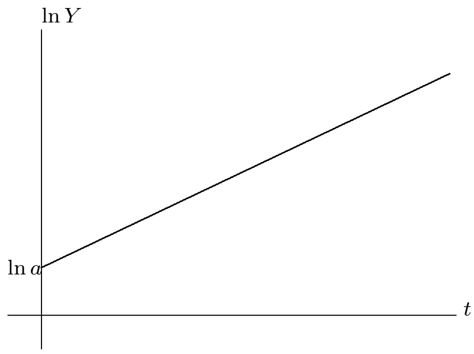
The equation relating Y and t is:

$$Y = ae^{0.03t}$$

Take logs of both sides of this equation:

$$\begin{aligned} \ln Y &= \ln(ae^{0.03t}) \\ &= \ln a + \ln(e^{0.03t}) \\ &= \ln a + 0.03t \end{aligned}$$

So if we plot a graph of $\ln Y$ against t, we will get a straight line, with gradient 0.03, and vertical intercept $\ln a$.



Exercises 4.11: **Economic Example using the Exponential Function**

The percentage of a firm's workforce who know how to use its computers increases over time according to:

$$P = 100\left(1 - e^{-0.5t}\right)$$

where t is the number of years after the computers are introduced.

(1) Calculate P for $t = 0, 1, 5$ and 10, and hence sketch the graph of P against t.

(2) What happens to P as $t \to \infty$?

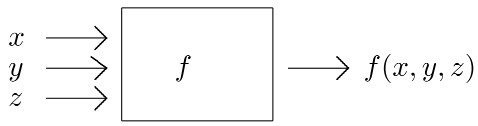
(3) How long is it before 95% of the workforce know how to use the computers?
Hint: You can use the method that we used in the first example, but you will need to rearrange your equation before you take logs of both sides.

Further reading and exercises

- *Jacques* §2.4.

7. Functions of Several Variables

A function can have more than one input:



EXAMPLES 7.1: *Functions of Several Variables*

(i) For the function $F(x, y) = x^2 + 2y$
   (1) $F(3, 2) = 3^2 + 4 = 13$
   (2) $F(-1, 0) = (-1)^2 + 0 = 1$

(ii) The production function of a firm is: $Y(K, L) = 3K^{0.4}L^{0.6}$
In this equation, Y is the number of units of output produced with K machines and L workers.

   (1) How much output is produced with one machine and 4 workers?
   $Y(1, 4) = 3 \times 4^{0.6} = 6.9$

   (2) If the firm has 3 machines, how many workers does it need to produce 10 units of output?
   When the firm has 3 machines and L workers, it produces:

$$\begin{aligned} Y &= 3 \times 3^{0.4}L^{0.6} \\ &= 4.66L^{0.6} \end{aligned}$$

   So if it wants to produce 10 units we need to find the value of L for which:

$$\begin{aligned} 10 &= 4.66L^{0.6} \\ \Longrightarrow L^{0.6} &= 2.148 \\ \Longrightarrow L &= 2.148^{\frac{1}{0.6}} \\ &= 3.58 \end{aligned}$$

   It needs 3.58 (possibly 4) workers.[1]

EXERCISES 4.12: **Functions of Several Variables**

(1) For the function $f(x, y, z) = 2x + 6y - 7 + z^2$ evaluate (i) $f(0, 0, 0)$ (ii) $f(5, 3, 1)$

(2) A firm has production function: $Y(K, L) = 4K^2L^3$, where K is the number of units of capital, and L is the number of workers, that it employs. How much output does it produce with 2 workers and 3 units of capital? If it has 5 units of capital, how many workers does it need to produce 6400 units of output?

[1] If you are unsure about manipulating indices as we have done here, refer back to Chapter 1.

7.1. Drawing Functions of Two Variables: Isoquants

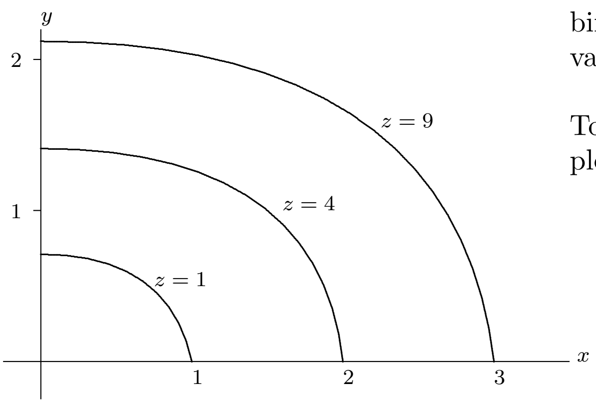
Suppose we have a function of two variables:

$$z = f(x, y)$$

To draw a graph of the function, you could think of (x, y) as a point on a horizontal plane, like the co-ordinates of a point on a map, and z as a distance above the horizontal plane, corresponding to the height of the land at the point (x, y). So the graph of the function will be a "surface" in 3-dimensions. See *Jacques* §5.1 for an example of a picture of a surface.

Since you really need 3 dimensions, graphs of functions of 2 variables are difficult to draw (and graphs of functions of 3 or more variables are impossible). But for functions of two variables we can draw a diagram like a contour map:

EXAMPLES 7.2: $z = x^2 + 2y^2$



The lines are *isoquants*: they show all the combinations of x and y that produce a particular value of the function, z.

To draw the isoquant for $z = 4$, for example:

- Write down the equation:
$$x^2 + 2y^2 = 4$$
- Make y the subject:
$$y = \sqrt{\frac{4 - x^2}{2}}$$
- Draw the graph of y against x.

7.2. Economic Application: Indifference curves

Suppose that a consumer's preferences over 2 goods are represented by the utility function:

$$U(x_1, x_2) = x_1^3 x_2^2$$

In this case an isoquant is called an *indifference curve*. It shows all the bundles (x_1, x_2) that give the same amount of utility: k, for example. So an indifference curve is represented by the equation:

$$x_1^3 x_2^2 = k$$

where k is a constant. To draw the indifference curves, make x_2 the subject:

$$x_2 = \left(\frac{k}{x_1^3}\right)^{\frac{1}{2}}$$

Then you can sketch them for different values of k (putting x_2 on the vertical axis).

EXERCISES 4.13: Sketch the indifference curves for a consumer whose preferences are represented by the utility function: $U(x_1, x_2) = x_1^{0.5} + x_2$

Further reading and exercises

- *Jacques* §5.1.
- *Anthony & Biggs* §11.1.

8. Homogeneous Functions, and Returns to Scale

EXERCISES 4.14: A firm has production function $Y(K,L) = 4K^{\frac{1}{3}}L^{\frac{2}{3}}$.

(1) How much output does it produce with

(1) 2 workers and 5 units of capital?

(2) 4 workers and 10 units of capital?

(3) 6 workers and 15 units of capital?

In this exercise, you should have found that when the firm doubles its inputs, it doubles its output, and when it trebles the inputs, it trebles the output. That is, the firm has constant return to scale. In fact, for the production function

$$Y(K,L) = 4K^{\frac{1}{3}}L^{\frac{2}{3}}$$

if the inputs are multiplied by any positive number λ:

$$\begin{aligned} Y(\lambda K, \lambda L) &= 4(\lambda K)^{\frac{1}{3}}(\lambda L)^{\frac{2}{3}} \\ &= 4\lambda^{\frac{1}{3}}K^{\frac{1}{3}}\lambda^{\frac{2}{3}}L^{\frac{2}{3}} = \lambda^{\frac{1}{3}+\frac{2}{3}}4K^{\frac{1}{3}}L^{\frac{2}{3}} = \lambda 4K^{\frac{1}{3}}L^{\frac{2}{3}} \\ &= \lambda Y(K,L) \end{aligned}$$

– the output is multiplied by λ. We say that the function is *homogeneous of degree* 1.

> A function $f(x,y)$ is said to be homogeneous of degree n if:
> $$f(\lambda x, \lambda y) = \lambda^n f(x,y)$$
> for all positive numbers λ.

We can extend this for functions of more than 2 variables in the obvious way.

EXAMPLES 8.1: *Homogeneous Functions*

(i) $f(x,y,z) = 2x^{\frac{1}{2}} + 3y^{\frac{1}{2}} + z^{\frac{1}{2}}$
For this function:

$$\begin{aligned} f(\lambda x, \lambda y, \lambda z) &= 2(\lambda x)^{\frac{1}{2}} + 3(\lambda y)^{\frac{1}{2}} + (\lambda z)^{\frac{1}{2}} \\ &= 2\lambda^{\frac{1}{2}}x^{\frac{1}{2}} + 3\lambda^{\frac{1}{2}}y^{\frac{1}{2}} + \lambda^{\frac{1}{2}}z^{\frac{1}{2}} \\ &= \lambda^{\frac{1}{2}}\left(2x^{\frac{1}{2}} + 3y^{\frac{1}{2}} + z^{\frac{1}{2}}\right) = \lambda^{\frac{1}{2}}f(x,y,z) \end{aligned}$$

So it is homogeneous of degree $\frac{1}{2}$.

(ii) The function $g(x_1,x_2) = x_1 + x_2^3$ is not homogeneous because:

$$g(\lambda x_1, \lambda x_2) = \lambda x_1 + (\lambda x_2)^3 = \lambda x_1 + \lambda^3 x_2^3$$

which cannot be written in the required form.

(iii) The production function $F(K, L) = 3KL$ is homogeneous of degree 2:

$$F(\lambda K, \lambda L) = 3\lambda K \lambda L = \lambda^2 F(K, L)$$

This means that if $K$and L are *increased* by the same factor $\lambda > 1$, then output increases by more:

$$F(\lambda K, \lambda L) = \lambda^2 F(K, L) > \lambda F(K, L)$$

So this production function has *increasing returns to scale.*

From examples like this we can see that:

A production function that is homogeneous of degree n has:
constant returns to scale if $n = 1$
increasing returns to scale if $n > 1$
decreasing returns to scale if $n < 1$

EXERCISES 4.15: **Homogeneous Functions**

(1) Determine whether each of the following functions is homogeneous, and if so, of what degree:
   (a) $f(x, y) = 5x^2 + 3y^2$
   (b) $g(z, t) = t(z + 1)$
   (c) $h(x_1, x_2) = x_1^2(2x_2^3 - x_1^3)$
   (d) $F(x, y) = 8x^{0.7}y^{0.9}$

(2) Show that the production function $F(K, L) = aK^c + bL^c$ is homogeneous. For what parameter values does it have constant, increasing and decreasing returns to scale?

Further reading and exercises

- *Jacques* §2.3.
- *Anthony & Biggs* §12.4.

Solutions to Exercises in Chapter 4

EXERCISES 4.1:

(1) $f(2) = 5$, and $f(-4) = 17$
(2) $g(x) = 0 \Rightarrow x = 1$
(3) $C(4) = 44$

EXERCISES 4.2:

(1) (a) $f(x) = x(x-1)(x-2) \Rightarrow f(0) = 0, f(1) = 0, f(2) = 0$
 (b) $f(x) < 0,\ x < 0$
 $f(x) > 0,\ 0 < x < 1$
 $f(x) < 0,\ 1 < x < 2$
 $f(x) > 0,\ x > 2$
(2) (a) 4
 (b) $g(x) = (x^2-1)(4-x^2) \Rightarrow$
 $g(-2) = 0$
 $g(-1) = 0$
 $g(1) = g(2) = 0$
 (c) $x < -2 : g(x) < 0$
 $-2 < x < -1 : g(x) > 0$
 $-1 < x < 1 : g(x) < 0$
 $1 < x < 2 : g(x) > 0$
 $x > 2 : g(x) < 0$

EXERCISES 4.3:

EXERCISES 4.4:

(1) (a) $\lim_{x\to\infty} f(x) = -\infty$
 (b) $\lim_{x\to-\infty} f(x) = -\infty$
 (c) $\lim_{x\to 0} f(x) = 2$
(2) Decreasing, $\lim_{x\to\infty} y = 3$
(3) (a) Increasing function.
 (b) $\lim_{x\to\infty} g(x) = 1$
 (c) $\lim_{x\to 0} g(x) = -\infty$

EXERCISES 4.5:

(1) $f(g(2)) = -27$, $g(f(2)) = -9$
(2) $g(h(1)) = \frac{4}{3}$, $h(g(1)) = 5$
(3) $k(m(3)) = 3$, $m(k(3)) = 3$. *Cube root* is the inverse of *cube*.
(4) $g(f(x)) = 2(x+1)^2$
 $f(g(x)) = 2x^2 + 1$
(5) $h(k(x)) = \frac{5x}{2x+1}$, $k(h(x)) = \frac{x+2}{5}$

EXERCISES 4.6:

(1) $f^{-1}(y) = \frac{y-7}{8}$
(2) $g^{-1}(y) = 6 - 2y$
(3) $h^{-1}(y) = \frac{1-4y}{y} = \frac{1}{y} - 4$
(4) $k^{-1}(y) = \sqrt[3]{y}$

EXERCISES 4.7:

(1) $P^S = \frac{Q+10}{6}$, $P^D = \frac{100}{Q}$
(2) $Q = 20$, $P = 5$

EXERCISES 4.8:

(1)
(2) $Q = \frac{a-d}{c+1}$, $P = \frac{ac+d}{c+1}$.
(3) $d > a \Rightarrow$ there is no equilibrium in the market.

EXERCISES 4.9:

(1)
(2) (a) 1
 (b) 1

EXERCISES 4.10:

(1)
(2) (a) 0
 (b) 1
 (c) $5x$
 (d) x^2
 (e) $3x$

EXERCISES 4.11:

(1) $t = 0,\ P = 0$
 $t = 1,\ P \approx 39.35$
 $t = 5,\ P \approx 91.79$
 $t = 10,\ P \approx 99.33$
(2) $\lim_{t\to\infty} P = 100$
(3) $t \approx 6$

EXERCISES 4.12:

(1) $f(0,0,0) = -7$, $f(5,3,1) = 22$
(2) $Y(3,2) = 288$, $L = 4$

EXERCISES 4.13:

EXERCISES 4.14:

(1) $Y(5,2) \approx 10.86$
(2) $Y(10,4) \approx 21.72$
(3) $Y(15,6) \approx 32.57$

EXERCISES 4.15:

(1) (a) 2
 (b) No
 (c) 5
 (d) 1.6
(2) $F(\lambda K, \lambda L) = a\lambda^c K^c + b\lambda^c L^c = \lambda^c F(K,L)$.
 Increasing $c > 1$, constant $c = 1$, decreasing $c < 1$

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[2]This Version of Workbook Chapter 4: September 25, 2014

Worksheet 4: Functions

Quick Questions

(1) If $f(x) = 2x - 5$, $g(x) = 3x^2$ and $h(x) = \frac{1}{1+x}$:
  (a) Evaluate: $h\left(\frac{1}{3}\right)$ and $g(h(2))$
  (b) Solve the equation $h(x) = \frac{3}{4}$
  (c) Find the functions $h(f(x))$, $f^{-1}(x)$, $h^{-1}(x)$ and $f(g(x))$.

(2) A country's GDP grows according to the equation $Y(t) = Y_0e^{gt}$.
(Y_0 and g are parameters; g is the growth rate.)
  (a) What is GDP when $t = 0$?
  (b) If $g = 0.05$, how long will it take for GDP to double?
  (c) Find a formula for the time taken for GDP to double, in terms of the growth rate g.

(3) Consider the function: $g(x) = 1 - e^{-x}$.
  (a) Evaluate $g(0)$, $g(1)$ and $g(2)$.
  (b) Is it an increasing or a decreasing function?
  (c) What is $\lim_{x\to\infty} g(x)$?
  (d) Use this information to sketch the function for $x \geq 0$.

(4) The inverse supply and demand functions for a good are: $P^s(Q) = 1 + Q$ and $P^d(Q) = a - bQ$, where a and b are parameters. Find the equilibrium quantity, in terms of a and b. What conditions must a and b satisfy if the equilibrium quantity is to be positive?

(5) Are the following functions homogeneous? If so, of what degree?
(a) $g(z, t) = 2t^2z$ (b) $h(a, b) = \sqrt[3]{a^2 + b^2}$

Longer Questions

(1) The supply and demand functions for beer are given by:

$$q^s(p) = 50p \quad \text{and} \quad q^d(p) = 100\left(\frac{12}{p} - 1\right)$$

  (a) How many bottles of beer will consumers demand if the price is 5?
  (b) At what price will demand be zero?
  (c) Find the equilibrium price and quantity in the market.
  (d) Determine the inverse supply and demand functions $p^s(q)$ and $p^d(q)$.
  (e) What is $\lim_{q\to\infty} p^d(q)$?
  (f) Sketch the inverse supply and demand functions, showing the market equilibrium.

The technology for making beer changes, so that the unit cost of producing a bottle of beer is 1, whatever the scale of production. The government introduces a tax on the production of beer, of t per bottle. After these changes, the demand function remains the same, but the new inverse supply function is:

$$p^s(q) = 1 + t$$

(g) Show the new supply function on your diagram.
(h) Find the new equilibrium price and quantity in the market, in terms of the tax, t.
(i) Find a formula for the total amount of tax raised, in terms of t, and sketch it for $0 \leq t \leq 12$. Why does it have this shape?

(2) A firm has m machines and employs n workers, some of whom are employed to maintain the machines, and some to operate them. One worker can maintain 4 machines, so the number of production workers is $n - \frac{m}{4}$. The firm's produces:

$$Y(n, m) = \left(n - \frac{m}{4}\right)^{\frac{2}{3}} m^{\frac{1}{3}}$$

units of output provided that $n > \frac{m}{4}$; otherwise it produces nothing.

(a) Show that the production function is homogeneous. Does the firm have constant, increasing or decreasing returns to scale?

(b) In the short-run, the number of machines is fixed at 8.
   (i) What is the firm's short-run production function $y(n)$?
   (ii) Sketch this function. Does the firm have constant, increasing, or decreasing returns to labour?
   (iii) Find the inverse of this function and sketch it. Hence or otherwise determine how many workers the firm needs if it is to produce 16 units of output.

(c) In the long-run, the firm can vary the number of machines.
   (i) Draw the isoquant for production of 16 units of output.
   (ii) What happens to the isoquants when the number of machines gets very large?
   (iii) Explain why this happens. Why would it not be sensible for the firm to invest in a large number of machines?